St. Andrew The First-called Georgian University of The Patriarchate of Georgia
Faculty of Natural Sciences

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# Virtual photons in ChPT: an arbitrary covariant gauge

by

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# Abstract

Chiral pertubation theory (ChPT) is regarded as an effective theory of quantum chromodynamics (QCD) at low-energies. The main goal of presented thesis was to investigate the gauge dependence of the one-loop generating functional for the mesons and virtual photons in the framework of ChPT. The motivation for the study was that it may provide a better understanding of the low-energy effective theory of QCD in presence of electromagnetic interaction. The divergent part of the aforementioned one-loop generating functional as well as the $\beta$-functions of the electromagnetic low-energy constants in the two-flavor case were calculated in an arbitrary covariant gauge. Comparison of the $\beta$-functions with the ones available in the literature was made. The independence of various physical quantities on the renormalization scale, as well as relations between two- and three-flavor $\beta$-functions was verified.

# 1 Introduction

Chiral perturbation theory (ChPT) is an effective field theory constructed with a Lagrangian consistent with the (approximate) chiral symmetry of quantum chromodynamics (QCD), as well as the other symmetries of parity and charge conjugation. ChPT is a theory which allows one to study the low-energy dynamics of QCD. As QCD becomes non-perturbative at low energy, it is impossible to use perturbative methods to obtain sensible theoretical predictions.

In the low-energy regime of QCD, the degrees of freedom are no longer quarks and gluons, but rather hadrons ( baryons and mesons ). This is a result of confinement. If one could "solve" the generating functional of QCD , (such that the degrees of freedom in the Lagrangian are replaced by hadrons) then one could extract information about low-energy physics. To date this has not been accomplished. A low-energy effective theory with hadrons as the fundamental degrees of freedom is a possible solution. The Lagrangian of effective theory contains all terms consistent with the symmetries of the underlying theory. In general there are an infinite number of terms which meet this requirement. Therefore in order to make any physical predictions, one assigns the theory a power counting scheme which organizes terms by a pre-specified degree of importance which allows one to keep some terms and reject all others as higher-order corrections which can be safely neglected. In addition, unknown coupling constants, also called low-energy constants , are associated with terms in the Lagrangian that must be determined by fitting to experimental data. In what follows we will only focus on mesonic sector of ChPT.

We have just mentioned the so-called strong sector of ChPT. However, that is not the whole story, since there exist electrically charged mesons. Thus, it is necessary to take into account electromagnetic corrections. The underlying theory, QCD+QED, depends on the strong coupling constant $g$, the fine structure constant $\alpha \approx 1/137$ and the light quark masses. The corresponding effective theory (ChPT with virtual photons) is based on expansion in powers of the electromagnetic coupling $e$. In addition, for consistency, one should provide chiral counting scheme, which ensures the renormalizability of the effective theory order by order.

In present thesis we consider the ChPT with virtual photons at one-loop level (next-to-leading order). In the first part (Sec.2) we briefly review the symmetry properties of strong interactions at low energies. Next, in the Sec.3 we show how to construct ChPT for mesons at leading and next-to-leading orders. In the Sec.4 we describe the inclusion of virtual photons in the ChPT framework up to one-loop. Then we consider the so-called one-loop generating functional. This object contains divergences, coming from the loops, in which mesons and photons run.These divergences can be absorbed by suitable renormalization of the low-energy constants, presented in the next-to-leading order Lagrangian.

The main topic of the thesis is a calculation of the divergent part of one-loop generating functional, as well as $\beta$-functions of the low-energy constants in arbitrary covariant gauge $a$. To date such calculation is presented in the literature only for the Feynman gauge $a$=1. Knowing of the explicit dependence on gauge parameter may provide a better understanding of the low-energy structure of the QCD+QED as well as it may be helpful for the extraction of the so-called electromagnetic low-energy constants from the lattice QCD data. Note that the one-loop generating functional is directly related to the differential operator $D$, determined in Sec. 4.3. In case of $a = 1$, the operator $D$ is of so-called "minimal" type and one can apply heat-kernel method to find the divergent part of generating functional. That is not the case for an arbitrary gauge $a \neq 1$. Therefore, we use alternative approach, considered in details in the Sec.4.4. Finally, we make a number of checks of obtained result.

# 2 The strong interactions at low energies

## 2.1 The symmetries of Quantum Chromodynamics

Quantum Chromodynamics (QCD) is a theory that describes strong interactions between quarks (matter fields) and gluons (gauge bosons)[1, 2, 3] making up hadrons (such as the proton, neutron or pion). It is a gauge theory which means that QCD Lagrangian should be invariant under a continuous group of local (gauge) transformations. The latter statement is known as gauge principle and it has proven to be a successful method in elementary particle physics to generate interactions between matter fields through the exchange of massless gauge bosons. QCD underlying gauge group is color SU(3). Quarks are spin-1/2 fermions, with six different flavors ( $u$,$d$,$s$,$c$,$b$,$t$) in addition to their three possible colors. Thus, each quark field $q_f$ ( subscript $f$ denotes quark flavor) has a form of a color triplet

$$q_f = \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix}$$

The SU(3) gauge invariant QCD Lagrangian has the following form

$$\mathcal{L}_{\text{QCD}} = \sum_{f=\substack{u,d,s,\\c,b,t}} \bar{q}_f(i\not{D} - m_f)q_f - \frac{1}{4}\, G_{\mu\nu,a}\, G_a^{\mu\nu}. \tag{1}$$

where $G_{\mu\nu,a}$ denotes gauge field strength tensor:

$$G_{\mu\nu,a} = \partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} + g f_{abc} A_{\mu,b} A_{\nu,c}$$

Here $A_{\mu,a}$ is a gauge potential (gluon field); $g$ coupling constant between quarks and gluons and $f_{abc}$ are structure constants of SU(3). The covariant derivative $D_\mu$ of Eq.(1) contains eight independent gauge fields $A_{\mu,a}$ and reads

$$D_\mu q_f = \partial_\mu q_f - ig \sum_{a=1}^{8} \frac{\lambda_a^C}{2} A_{\mu,a} q_f;$$

here $\lambda_a^C$ are Gell-Mann matrices, wich act in color space. The existence of one universal coupling constant $g$ means that interaction between quarks and gluons is independent of the quarks flavors.Moreover, as SU(3) is a non-Abelian Lie group, the entire theory is non-linear, i.e., the gluons interact with each other.

The values of light ( $u$,$d$,$s$ ) and heavy ($c$,$b$,$t$ ) flavors masses can be arranged on a typical hadronic scale $\Lambda_{hadr} = 1\text{GeV}$ as follows

$$\begin{pmatrix} m_u = 0.005\,\text{GeV} \\ m_d = 0.009\,\text{GeV} \\ m_s = 0.175\,\text{GeV} \end{pmatrix} \ll 1\,\text{GeV} \leq \begin{pmatrix} m_c = (1.15 - 1.35)\,\text{GeV} \\ m_b = (4.0 - 4.4)\,\text{GeV} \\ m_t = 174\,\text{GeV} \end{pmatrix}, \tag{2}$$

Note that there are presented so-called current-quark masses, which must not be confused with the constituent quark masses of a nonrelativistic quark model which are of the order 0.35 GeV.

Above a typical hadronic mass scale of about 1 GeV, there is a large number of states, both meson resonances and baryons. Only a very few (pseudoscalar) states, however, are significantly lighter than $\Lambda_{hadr} = 1\text{GeV}$: in particular the pions ($M_\pi \approx 140$ MeV), but also kaons ($M_K \approx 495$ MeV) and the eta ($M_\eta \approx 550$ MeV).

The masses of the lightest meson and baryon containing a charmed quark ($c$ quark), $D^+ = c\bar{d}$ and $\Lambda_c^+ = udc$, are $(1869.4 \pm 0.5)\,\text{MeV}$ and $(2284.9 \pm 0.6)\,\text{MeV}$, respectively. The threshold center-of-mass energy to produce a $D^+D^-$ pair in $e^+e^-$ collisions is approximately 3.74 GeV $>$ 1GeV. Since we are interested in a low-energy regime ( <1Gev ) one can neglect heavy quarks contributions and consider the part of QCD Lagrangian, containing only light flavors. In addition, as one can conclude from Eq.(2), light quark masses are much smaller than hadronic scale. Therefore, as a good approximation to describe low-energy QCD one can consider QCD Lagrangian in the so-called chiral limit $m_u, m_d, m_s \to 0$:

$$\mathcal{L}^0_{qcd} = \sum_{l=u,d,s} \bar{q}_l i \not{D}\, q_l - \frac{1}{4} G_{\mu\nu,a} G^{\mu\nu}_a. \tag{3}$$

The Lagrangian $\mathcal{L}^0_{qcd}$ in chiral limit, apart from gauge symmetry,Lorentz invariance and the discrete symmetries $P$,$C$,$T$, has additional symmetry, called chiral symmetry. To investigate this symmetry, we decompose the quark fields into its chiral components according to

$$q = \frac{1}{2}(1-\gamma_5)q + \frac{1}{2}(1+\gamma_5)q = P_L q + P_R q \;=\; q_L + q_R\ ;$$

here $P_R = P_R^\dagger$ and $P_L = P_L^\dagger$ are correspondingly right-handed ($R$) and left-handed ($L$) projection operators. They satisfy a completeness relation

$$P_R + P_L = 1,$$

are idempotent,

$$P_R^2 = P_R, \quad P_L^2 = P_L,$$

and respect the orthogonality relations

$$P_R P_L = P_L P_R = 0.$$

Using above relations, we can write QCD Lagrangian in the chiral limit:

$$\mathcal{L}^0_{qcd} = \sum_{l=u,d,s} (\bar{q}_{R,l} i\not{D}\, q_{R,l} + \bar{q}_{L,l} i\not{D}\, q_{L,l}) - \frac{1}{4} G_{\mu\nu,a} G^{\mu\nu}_a. \tag{4}$$

It is invariant under global chiral $U(3)_L \times U(3)_R$ transformations

$$q_R \mapsto U_R q_R, \quad q_L \mapsto U_L q_L,$$

where $U_L$ and $U_R$ are independent unitary $3 \times 3$ matrices:

$$U_R = \exp\left(-i \sum_{a=1}^{8} \Theta^R_a \frac{\lambda_a}{2}\right) e^{-i\Theta^R}, \quad U_L = \exp\left(-i \sum_{a=1}^{8} \Theta^L_a \frac{\lambda_a}{2}\right) e^{-i\Theta^L};$$

here $\Theta^{R,L}_a$ $(a = 1, ..., 8)$ and $\Theta^{R,L}$ are group parameters.

We can rewrite the symmetry group according to

$$\mathrm{U}(3)_L \times \mathrm{U}(3)_R = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A \tag{5}$$

The Noether's theorem states that the consequence of global symmetry is the existence of conserved currents. Let's consider currents, associated with the global symmetry transformations, given by Eq.(5) (see [4] for details ).

- The $U(1)_V$ singlet vector current $V^\mu = \bar{q}\gamma^\mu q$,also called the quark number or baryon number, is conserved in the Standard Model.
- The $U(1)_A$ axial-vector singlet current $A^\mu = \bar{q}\gamma^\mu\gamma_5 q$ is no more conserved due to quantum effects, referred to as $U(1)_A$ anomaly.
- The $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ conserved chiral currents

$$V^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q, \quad A^{\mu,a} = \bar{q}\gamma^\mu\gamma_5 \frac{\lambda^a}{2} q, a = 1, \ldots, 8$$

Therefore, we are left with the invariance of the Lagrangian $\mathcal{L}^0_{qcd}$ under global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ transformations.

We can associate with any conserved current $J^{\mu,a}$, $\partial_\mu J^{\mu,a} = 0$ time-independent quantity, called charge

$$Q^a(t) = \int d^3x J^a_0(t, \vec{x})$$

In our case charge operators

$$Q^a_V(t) = \int dx\, V^{0,a}(x), \quad Q^a_A(t) = \int dx\, A^{0,a}(x), \quad Q_V(t) = \int dx\, V^0(x)$$

form the Lie algebra of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ group [4]:

$$[Q_V^a, Q_V^a] = if_{abc}Q_V^c, \quad [Q_A^a, Q_A^a] = if_{abc}Q_V^c, \quad [Q_V^a, Q_A^a] = if_{abc}Q_A^c,$$

$$[Q_V^a, Q_V] = [Q_A^a, Q_V] = 0$$

In reality, $u-$, $d-$, and $s-$quark masses are finite and quark-mass term in the QCD Lagrangian mixes left- and right-handed fields:

$$\mathcal{L}_M = -\bar{q}Mq = -(\bar{q}_R M q_L + \bar{q}_L M q_R). \tag{6}$$

where

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

is quark-mass matrix. The quark-mass term explicitly breaks chiral symmetry of the QCD Lagrangian and corresponding currents are no more conserved. More precisely, the divergences of the currents read [4]

$$\begin{aligned} \partial_\mu V^{\mu,a} &= i\bar{q}[M, \frac{\lambda_a}{2}]q, \\ \partial_\mu A^{\mu,a} &= i\bar{q}\{\frac{\lambda_a}{2}, M\}\gamma_5 q, \\ \partial_\mu V^{\mu} &= 0, \\ \partial_\mu A^{\mu} &= 2i\bar{q}M\gamma_5 q + \frac{3g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}G_a^{\mu\nu}G_a^{\rho\sigma}, \quad \epsilon_{0123} = 1, \end{aligned} \tag{7}$$

where the last term in the divergence of the singlet axial-vector current accounts for the already mentioned axial anomaly. Note that the divergences of the eight axial-vector current of Eq.(7) are proportional to pseudoscalar quadratic forms. Due to smallness of the light quarks masses on a typical hadronic scale 1 GeV this divergences are expected to be a small and one can interpret them as the origin of the PCAC relation (partially conserved axial-vector current) [5, 6].

## 2.2 Spontaneous Symmetry Breaking in QCD

We saw in previous subsection that the light-flavor QCD Lagrangian is invariant under group $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$. Now, we have to investigate whether chiral symmetry is realised in Nature in the Wigner-Weyl mode, i.e. the symmetry is manifest in the spectrum in terms of multiplets, or whether it is realised as a Goldstone mode, i.e the symmetry is hidden or spontaneously broken. A continuous symmetry is said to be spontaneously broken or hidden, if the ground state of the system is no longer invariant under the full symmetry group of the Hamiltonian.

In the case of Wigner-Weyl mode the conserved axial charges $Q_A^a$ annihilating the vacuum,

$$Q_A^a|0\rangle = 0,$$

would lead to parity doubling in the hadron spectrum. However, no such degenerate multiplets with opposite (negative) parity are observed experimentally. Phenomenologically, there are (approximate) $SU(3)_V$ multiplets. In addition,unbroken chiral symmetry would lead to a vanishing difference of the vector–vector and axial–axial vacuum correlators, $\langle 0|VV|0\rangle - \langle 0|AA|0\rangle = 0$. This difference can be measured in hadronic tau decays $\tau \to \nu_\tau + n\,\pi$, leading to a non-vanishing result [7].

If chiral symmetry is realised in the Goldstone mode, then as it was shown in Ref. [8], the ground state is invariant only under subgroup of $G$ $H = \mathrm{SU}(3)_V \times \mathrm{U}(1)_V$ transformations, that is the charges $Q_V^a$ and $Q_V$ annihilate the ground state (vacuum):

$$Q_V^a|0\rangle = Q_V|0\rangle = 0.$$

According to Coleman's theorem [9], if the vacuum is invariant under $\mathrm{SU}(3) \times U(1)_V$, then so is the Hamiltonian (but not vice versa). This further implies that the physical states of the spectrum of the QCD Hamiltonian $H^0_{qcd}$ can be organized according to irreducible representations of $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$. The index $V$ indicates that the generators transform with a positive sign under parity. The $\mathrm{U}(1)_V$ symmetry results in baryon number conservation and leads to a classification of hadrons into mesons (B = 0) and baryons (B = 1). Then, since the parity doubling is not observed for the low-lying states, one assumes that the $Q_A^a$ do not annihilate the ground state:

$$Q_A^a|0\rangle \neq 0.$$

Thus, the $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry spontaneously breaks down to $\mathrm{SU}(3)_V$:

$$\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \xrightarrow{\mathrm{SSB}} \mathrm{SU}(3)_V\ .$$

According to Goldstone's theorem [10, 11], to each axial generator $Q_A^a$, which does not annihilate the ground state, corresponds a massless Goldstone boson field $\phi^a(x)$ with spin 0, whose symmetry properties are closely connected to the generator in question. In particular, the Goldstone bosons are pseudoscalars, which means that they transform under parity as

$$\phi^a(t,\vec{x}) \overset{P}{\mapsto} -\phi^a(t,-\vec{x}) \tag{8}$$

Also, they transform under the subgroup $\mathrm{SU}(3)_V$ as an octet:

$$[Q_V^a, \phi^b(x)] = i f_{abc}\phi^c(x)$$

Since there are eight broken axial generators of the chiral group, $Q_A^a$, there should be eight pseudoscalar Goldstone states. The experiment shows that the full octet of pseudoscalar mesons (lightest hadronic states) $\pi^{\pm}$, $\pi^0$, $K^{\pm}$, $K^0$, $\bar{K}^0$, and $\eta$, indeed carry the quantum numbers of Goldstone bosons. However, if pseudoscalar mesons were effectively Goldstone bosons, they generated by the spontaneous breakdown would had been massless. That is not the case in "real" world because the quark masses explicitly break the symmetry, but since $m_{u,d,s} < \Lambda_\chi \approx$1GeV the breaking is expected to be small enough and can be treated as small perturbation.Here the breaking scale of chiral symmetry $\Lambda_\chi$ plays an important role in construction of the effective theory of QCD at low energies. A motivation for such effective theory,called Chiral Perturbation Theory (ChPT) is that at low energies perurbative approach in the QCD is no longer applicable, since coupling constant $g$ becomes too large at energies below 1GeV.

In addition, we would like to mention theoretical conditions for a spontaneous chiral symmetry breaking in QCD [4]. Firstly, a non-vanishing scalar quark condensate, which is the quantity $\langle 0|\bar{q}q|0\rangle$ is a sufficient but not a necessary condition for a spontaneous chiral symmetry breakdown in QCD:

$$\langle 0|\bar{q}q|0\rangle \neq 0$$

Secondly, considering the nonzero matrix element of the axial-vector current between the vacuum and massless one particle states $|\phi^b\rangle$, which because of Lorentz covariance can be written as

$$\langle 0|A_\mu^a(0)|\phi^b(p)\rangle = ip_\mu F_0 \delta^{ab},$$

one obtains that nonzero value of $F_0$ (this constant will be introduced again later) is a necessary and sufficient criterion for spontaneous chiral symmetry breaking.

# 3 ChPT for mesons

## 3.1 Transition from QCD to ChPT

We mentioned that the interactions between quarks and gluons, ruled by QCD, are highly non-perturbative at energies below the breaking scale of chiral symmetry $\Lambda_\chi \approx$1GeV. This makes very difficult any description of the low-energy hadronic world in terms of quark and gluons. On the other hand it is experimental fact that the low-energy spectrum of the theory contains only octet of light pseudoscalar mesons $(\pi, K, \eta)$ and they interact weakly, both among themselves and with nucleons. We can expect that in terms of observable at low energies hadronic degrees of freedom it is possible to construct such an effective feld theory that makes possible to analyse the low energy structure of QCD.

The theoretical basis, which determined successful application of such effective field theories was formulated by Weinberg [12]. It boils down to the following statement (conjecture):
*Quantum Field Theory has no content besides unitarity, analyticity, cluster decomposition, and symmetries.*
This means that in order to calculate the S-matrix for any theory below some scale, one uses the most general effective Lagrangian consistent with these principles in terms of the appropriate asymptotic states. We will follow this principle in the construction of an effective theory for the strong interactions.

Chiral perturbation theory (ChPT) provides a systematic method for discussing the consequences of the global flavor symmetries of QCD at low energies by means of an effective field theory.At quite low energies, the corresponding Lagrangian is expressed in terms of the members of octet of light pseudoscalar mesons ($\pi^+, \pi^-, \pi^0, \eta, K^+, K^-, K^0$ and $\bar{\mathrm{K}}^0$). Such effective field theory is called the ChPT for mesons. We note that it is also possible to construct the ChPT for baryons (like protons and neutrons), but it is beyond the scope of this thesis.

In order to relate effective theory with underlying theory (QCD) let us consider generating functional of QCD in the presence of external fields. In order to do this, we equip the QCD Lagrangian with external fields (sources, [13], [14]) $v^\mu(x), v^\mu_{(s)}(x), a^\mu(x), s(x), p(x)$ coupled to the currents $V^{\mu,a}$,$V^\mu$, $A^{\mu,a}$ (see sec.1.1), as well as scalar $S = \bar{q}q$ and pseudoscalar $P = i\bar{q}\gamma^5 q$ densities:

$$\mathcal{L} = \mathcal{L}^0_{qcd} + \mathcal{L}_{ext} = \mathcal{L}^0_{qcd} + \bar{q}\gamma_\mu(v^\mu + \frac{1}{3}v^\mu_{(s)} + \gamma_5 a^\mu)q - \bar{q}(s - i\gamma_5 p)q. \quad (9)$$

Note that external fields are color-neutral Hermitian matrices:

$$v^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} v^\mu_a, \quad a^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} a^\mu_a, \quad s = \sum_{a=0}^{8} \lambda_a s_a, \quad p = \sum_{a=0}^{8} \lambda_a p_a.$$

Then the generating functional, wich is a vacuum-to-vacuum transition amplitude in the presence of external fields, has the form:

$$\exp[iZ(v,a,s,p)] = \langle 0;\mathrm{out}|0;\mathrm{in}\rangle_{v,a,s,p} = \langle 0|T\exp\left[i\int d^4x\mathcal{L}_{ext}(x)\right]|0\rangle$$
$$= \langle 0|T\exp\left(i\int d^4x\bar{q}(x)\{\gamma_\mu[v^\mu(x)+\gamma_5 a^\mu(x)] - s(x) + i\gamma_5 p(x)\}q(x)\right)|0\rangle, \tag{10}$$

The quark mass matrix $M=$diag$(m_u, m_d, m_s)$ is contaned in the scalar field $s(x)$. The Green functions formed with the current operators of massless QCD are obtained by expanding the generating functional around $v^\mu = v^\mu_{(s)} = a^\mu = s = p = 0$, whereas for the real world one has to expand around $v^\mu = v^\mu_{(s)} = a^\mu = p = 0, s(x) = M$. In the absence of anomalies, the Ward identities which express the symmetry properties of the theory in terms of the Green functions are equivalent to gauge invariance of the generating functional under local transformations of external fields [15].

The QCD Lagrangian $\mathcal{L}$ is invariant under local $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ transformations of the quark fields and external sources[4]:

$$\begin{aligned}
q_R &\mapsto \exp\left(-i\frac{\Theta(x)}{3}\right)V_R(x)q_R,\\
q_L &\mapsto \exp\left(-i\frac{\Theta(x)}{3}\right)V_L(x)q_L,\\
r_\mu &\mapsto V_R r_\mu V_R^\dagger + iV_R\partial_\mu V_R^\dagger,\\
l_\mu &\mapsto V_L l_\mu V_L^\dagger + iV_L\partial_\mu V_L^\dagger,\\
v_\mu^{(s)} &\mapsto v_\mu^{(s)} - \partial_\mu\Theta,\\
s+ip &\mapsto V_R(s+ip)V_L^\dagger,\\
s-ip &\mapsto V_L(s-ip)V_R^\dagger,
\end{aligned} \tag{11}$$

where $V_R(x)$ and $V_L(x)$ are independent space-time-dependent SU(3) matrices and $r_\mu = v_\mu + a_\mu, \quad l_\mu = v_\mu - a_\mu$.

Now, at the hadronic level generating functional is calculated with an effective Lagrangian $\mathcal{L}_{eff}$ but the with same external fields $v^\mu, v^\mu_{(s)}, a^\mu, p, s$:

$$\exp[iZ(v,a,s,p)] = \langle 0;\mathrm{out}|0;\mathrm{in}\rangle_{v,a,s,p} = \langle 0|T\exp\left[i\int d^4x\mathcal{L}_{eff}(x)\right]|0\rangle \tag{12}$$

This formula provides a link between underlying (QCD) and effective theory (ChPT). While the left-hand side represents the generating functional for the Green functions of the underlying theory, the right-hand side only involves the effective Lagrangian.

## 3.2 Construction of the effective Lagrangian for mesons

Since we are interested in processes were the momenta are small $q \ll \Lambda_\chi$ (the low energy sector of the theory), we can expand the Green functions in powers of the external momenta. This amounts to an expansion in derivatives of the external fields. However, the low energy expansion is not a simple Taylor expansion since the Goldstone bosons generate poles at $q^2 = 0$ (in the chiral limit) or $q^2 = M_\pi^2$(for finite quark masses, $M_\pi$ is the pion mass). The low energy expansion involves two small parameters, the external momenta $q$ and the quark masses $M$.Then, one expands in powers of these with the ratio $M/q^2$ fixed [14]. The low energy expansion of generating functional Eq.(10) is now obtained from a perturbative expansion of the ChPT Lagrangian:

$$\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + ...,$$

where the subscript (n=2,4) denotes the low energy dimension or so-called chiral order (number of derivatives and/or quark mass term). In other words, one can systematically approximate the underlying generating functional $Z_{QCD}(v, a, s, p)$ by a sequence:

$$Z_{QCD}(v, a, s, p) = Z_{eff}(v, a, s, p)^{(2)} + Z_{eff}(v, a, s, p)^{(4)} + ...,$$

where the generating functionals are obtained using the effective theory. Now, since the symmetry of effective theory, containing in the Ward identities is equivalent to gauge invariance of the generating functional one need to promote the global symmetry of the effective Lagrangian $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)$ to a local one [15]. While the external fields transform according to Eq.(11), the meson fields $\phi^a$, which we associate with the Goldstone bosons, transform with a nonlinear representation of G, spontaneously broken to $H = SU(3)_V$. Following the formalism developed in Ref. [16, 17] ( Callan, Coleman, Wess and Zumino or CCWZ formalism) and applying it to QCD, the meson fields are collected in a unitary matrix field $U(\phi)$ transforming as

$$U(\phi) \mapsto V_R U(\phi) V_L^\dagger, \qquad V_L(x) \in \mathrm{SU}(3)_L, \quad V_R(x) \in \mathrm{SU}(3)_R \tag{13}$$

under local chiral rotations $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$. There are different parameterizations of $U(\phi)$ corresponding to different choices of coordinates for the chiral coset space $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R/\mathrm{SU}(3)_V$. For convenience one chooses the matrix $U(x) \equiv U(\phi(x))$ to be the SU(3) matrix:

$$U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right),$$

where

$$\phi(x) \;=\; \sum_{a=1}^{8} \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \tag{14}$$

The local nature of G requires the introduction of a covariant derivative

$$d_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu,$$

$$d_\mu U \xrightarrow{\text{G}} V_R d_\mu U V_L^+$$

and of associated field strength tensors $f^L_{\mu\nu}$ and $f^R_{\mu\nu}$ corresponding to the external fiels $r_\mu$ and $l_\mu$,

$$\begin{aligned} f^R_{\mu\nu} &\equiv \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \\ f^L_{\mu\nu} &\equiv \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu]. \end{aligned}$$

Finally, we introduce the liner combination

$$\chi = 2B_0(s + ip),$$

with the scalar and pseudoscalar external fields [14]; $B_0$ is a constant which can be related to the quark condensate. Introduced quantities together with their transformation properties under the group (G), charge conjugation(C) and parity (P) (see Table 1, [4]) can be used for construction of the locally chiral invariant effective Lagrangian :

$$\mathcal{L}_{eff} = \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_4 + ...,$$

where due to Lorentz invariance only terms in even powers of derivatives occur. Note that since $U$ is unitary $UU^\dagger = I$, $\mathcal{L}_0$ can noly be a constant. Therefore ChPT Lagrangian for mesons has a form

$$\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + ..., \tag{15}$$

| element | $G$ | $C$ | $P$ |
|---|---|---|---|
| $U$ | $V_R U V_L^\dagger$ | $U^T$ | $U^\dagger$ |
| $d_{\lambda_1}\cdots d_{\lambda_n} U$ | $V_R d_{\lambda_1}\cdots d_{\lambda_n} U V_L^\dagger$ | $(d_{\lambda_1}\cdots d_{\lambda_n} U)^T$ | $(d^{\lambda_1}\cdots d^{\lambda_n} U)^\dagger$ |
| $\chi$ | $V_R \chi V_L^\dagger$ | $\chi^T$ | $\chi^\dagger$ |
| $d_{\lambda_1}\cdots d_{\lambda_n} \chi$ | $V_R d_{\lambda_1}\cdots d_{\lambda_n} \chi V_L^\dagger$ | $(d_{\lambda_1}\cdots d_{\lambda_n} \chi)^T$ | $(d^{\lambda_1}\cdots d^{\lambda_n} \chi)^\dagger$ |
| $r_\mu$ | $V_R r_\mu V_R^\dagger + iV_R \partial_\mu V_R^\dagger$ | $-l_\mu^T$ | $l^\mu$ |
| $l_\mu$ | $V_L l_\mu V_L^\dagger + iV_L \partial_\mu V_L^\dagger$ | $-r_\mu^T$ | $r^\mu$ |
| $f^R_{\mu\nu}$ | $V_R f^R_{\mu\nu} V_R^\dagger$ | $-(f^L_{\mu\nu})^T$ | $f_L^{\mu\nu}$ |
| $f^L_{\mu\nu}$ | $V_L f^L_{\mu\nu} V_L^\dagger$ | $-(f^R_{\mu\nu})^T$ | $f_R^{\mu\nu}$ |

Table 1: *Transformation properties of the building blocks under the group ($G$), charge conjugation ($C$), and parity ($P$). The expressions for adjoint matrices are obtained by taking the Hermitian conjugate of each entry.*

The $\mathcal{L}_2$ contains either two derivatives or one quark mass term. In other words $\mathcal{L}_2$,called leading-order Lagrangian, contains terms of the chiral order $O(p^2)$; $\mathcal{L}_4$ contains terms of chiral order $O(p^4)$ etc. To construct each term in $\mathcal{L}_{eff}$ building blocks should be counted as:

$$U = O(1),\ D_\mu U = O(p),\ r_\mu, l_\mu = O(p),\ f^R_{\mu\nu}, f^L_{\mu\nu} = O(p^2),\ \chi = O(p^2). \qquad (16)$$

The general scheme of construction of the $\mathcal{L}_{eff}$ in terms of building blocks of Eq.(16) can be outlined as follows [4]. Given objects $A, B, \ldots$, all of which transform as $A' = V_R A V_L^\dagger$, $B' = V_R B V_L^\dagger$, ..., one can form invariants by taking the trace of products of the type $AB^\dagger$:

$$\begin{aligned}\mathrm{Tr}(AB^\dagger) &\mapsto \mathrm{Tr}[V_R A V_L^\dagger (V_R B V_L^\dagger)^\dagger] = \mathrm{Tr}(V_R A V_L^\dagger V_L B^\dagger V_R^\dagger) = \mathrm{Tr}(AB^\dagger V_R^\dagger V_R) \\ &= \mathrm{Tr}(AB^\dagger).\end{aligned}$$

The generalization to more terms is straightforward; the product of invariant traces is invariant:

$$\mathrm{Tr}(AB^\dagger CD^\dagger), \quad \mathrm{Tr}(AB^\dagger)\mathrm{Tr}(CD^\dagger), \quad \cdots. \qquad (17)$$

## 3.3 The leading-order effective Lagrangian

One can apply this formalism to construct the most general, Lorentz,C,P and locally-invariant, effective Lagrangian at lowest chiral $O(p^2)$ [4, 13, 14]:

$$\mathcal{L}_2 = \frac{F_0^2}{4}\mathrm{Tr}[d_\mu U (d^\mu U)^\dagger] + \frac{F_0^2}{4}\mathrm{Tr}[\chi U^\dagger + U\chi^\dagger] = \frac{F_0^2}{4}\langle d_\mu U d^\mu U^\dagger + \chi U^\dagger + U\chi^\dagger\rangle, \qquad (18)$$

where it is assumed that $\langle \ldots \rangle \equiv \mathrm{Tr}[\ldots]$ and $d^\mu U^\dagger \equiv (d^\mu U)^\dagger$. Here $\mathcal{L}_2$ contains two free parameters, called low energy constants $F_0$ and $B_0$. Note that the $\mathcal{L}_2$ has the same for both $SU(3)$ and $SU(2)$. In order to determine the constant $F_0$ note that the Noether currents $V^{\mu,a}, A^{\mu,a}$ from $\mathcal{L}_2$ are given by

$$V^{\mu,a} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a[U, \partial^\mu U^\dagger]\right), \qquad (19)$$

$$A^{\mu,a} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a\{U, \partial^\mu U^\dagger\}\right). \qquad (20)$$

Then to find the leading term one should expand $A^{\mu,a}$ in the meson fields,

$$A^{\mu,a} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a\left\{1 + \cdots, -i\frac{\lambda_b \partial^\mu \phi_b}{F_0} + \cdots\right\}\right) = -F_0 \partial^\mu \phi_a + \cdots$$

such that we can calculate the matrix element of the axial current between a one-boson state and the vacuum,

$$\begin{aligned}\langle 0|A^{\mu,a}(x)|\phi^b(p)\rangle &= \langle 0| - F_0 \partial^\mu \phi_a(x)|\phi^b(p)\rangle \\ &= ip^\mu F_0 \exp(-ip\cdot x)\delta^{ab}.\end{aligned}$$

Thus, the $F_0$ can be identified with the pion (meson) decay constant(in the chiral limit), which is measured in pion decay $\pi^+ \to \ell^+\nu_\ell$ , $F_0 = F_\pi[1 + O(M)]$. The constant $B$, which appears in the field $\chi$, is related to the explicit symmetry breaking. One can choose $p = 0$ and $s = M$ ($\chi = 2B_0M$) and expand the symmetry breaking part of $\mathcal{L}_2$ in powers of the meson fields

$$\mathcal{L}_2^{SB} = \frac{1}{2}F_0^2 B\mathrm{Tr}[M(U + U^\dagger)] = (m_u + m_d + m_s)B[F_0^2 - \frac{\phi^2}{2} + \frac{\phi^4}{24F_0^2} + \cdots], \tag{21}$$

where the superscript SB refers to symmetry breaking. The first term in the right hand side of Eq.(21) is related to the vacuum energy, while the second and the third are meson mass and interaction terms, respectively.One can show that $B_0$ is proportional to vacuum expectation value of quark condensate:

$$\langle 0|\bar{q}q|0\rangle = -3F_0^2 B_0[1 + O(M)]. \tag{22}$$

Furthermore the meson masses, calculated from Eq.(21), in the case of isospin symmetry ($m_u = m_d = m$) are given by

$$\begin{aligned} M_\pi^2 &= 2mB_0[1 + O(M)], \\ M_K^2 &= (m + m_s)B_0[1 + O(M)], \\ M_\eta^2 &= \frac{2}{3}(m + 2m_s)\, B_0[1 + O(M)]. \end{aligned} \tag{23}$$

This results, in combination with Eq.(22) are referred to as the Gell-Mann, Oakes, and Renner relations [20].Furthermore, the masses of Eq.(23) satusfy the Gell-Mann-Okubo relation

$$4M_K^2 = 4B_0(m + m_s) = 2B_0(m + 2m_s) + 2B_0m = 3M_\eta^2 + M_\pi^2 \tag{24}$$

which is found to be fulfilled in nature to 7% accuracy. We see that quadratic masses of the Goldstone bosons linearly depend on the quark condensate and the quark masses. The latter result is supported by the analysis of the data on $K^+ \to \pi^+\pi^-e^+\nu_e$ [18][19], which means that the quark condensate really characterizes spontaneous chiral symmetry breaking in QCD.

One can now calculate tree diagrams using the effective Lagrangian $\mathcal{L}_2$ and derive all so called current algebra predictions (low energy theorems). Moreover, current algebra is only the first term in a systematic low energy expansion. Working out tree graphs using $\mathcal{L}_2$ can not be sufficient because the tree diagrams are always real and thus unitarity is violated. One has to include higher order corrections to deal with this problems. In order to do it in a consistent fashion, one needs a counting scheme to be discussed next.

## 3.4 Chiral counting scheme

So far we have only considered chiral Lagrangian for meson at leading order, i.e $O(p^2)$. As was already pointed out tree level contributions from $\mathcal{L}_2$ violates unitarity. Indeed, consider the pion-pion ($\pi\pi$) scattering to leading order. The scattering amplitude in the isospin limit $m_u = m_d$ can be decomposed as

$$M(\pi^a\pi^b \to \pi^c\pi^d) = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,u,s) + \delta^{ad}\delta^{bc}A(u,s,t) \ ,$$

where $u, s, t$ are so-called Mandelstam variables and $A(s,t,u)$ is invariant amplitude. One can calculate $A(s,t,u)$ from $\mathcal{L}_2$:

$$A(s,t,u) = \frac{s - M_\pi^2}{F^2} \ ,$$

a parameter-free prediction [21]. Note that $A(s,t,u)$ is *real*. However, the unitarity requires the partial waves $t_l^I$ to obey

$$\mathrm{Im}\, t_\ell^I = \sqrt{1 - \frac{4M_\pi^2}{s}}\left|t_\ell^I\right|^2 \ ;$$

here $I$ denotes the isospin $I = 0, 1, 2$ and $l$ is azimuthal quantum number $l = 0, 1, 2, \cdots$.The correct imaginary parts are only generated perturbatively by loops. Then the question arises whether it is possible to take into account loop correction in a consistent manner, such that one could calculate given matrix element with defined accuracy, using effective Lagrangian. It was shown in Ref. [12] that it is, indeed possible and corresponding rule, known as Weinberg's power counting scheme (or argument), have been formulated.

Consider an arbitrary loop diagram based on the general effective Lagrangian $\mathcal{L}_{eff} = \sum_n \mathcal{L}_n$, where $n$ denotes the chiral power of the various terms. Then the amplitude $\mathcal{A}$ of a diagram with $L$ loops , $I$ internal lines, and $V_n$ vertices of order $n$ behaves in term of powers of momenta as

$$\mathcal{A} \propto \int (d^4p)^L \frac{1}{(p^2)^I} \prod_n (p^n)^{V_n} \ . \tag{25}$$

Then let $\mathcal{A}$ be of chiral dimension $\mathcal{D} = 4L - 2I + \sum_n nV_n$. Using the topological identity $L = I - \sum_n V_n + 1$ to eliminate $L$ we find

$$\mathcal{D} = \sum_n V_n(n-2) + 2L + 2 \ . \tag{26}$$

Note since the chiral Lagrangian starts with $\mathcal{L}_2$, i.e. $n \geq 2$, the right-hand-side of Eq.(26) is a sum of non-negative terms. Consequently, for fixed $\mathcal{D}$, there is only a finite number of combination $L$, $V_n$ that can contribute and. In other words, only finite number of terms in the $\mathcal{L}_{eff}$ are needed to work to a fixed order in $p$, and the chiral Lagrangian acts like a renormaliazable field theory. Furthermore, each additional loop integration suppresses the amplitude by two orders in the momentum expansion. To illustrate this scheme, consider again $\pi\pi$ scattering. At $O(p^2)$, only tree level diagrams composed of vertices of $\mathcal{L}_2$ contribute ($V_{n>2} = 0, L = 0$) (see Fig.1(a), [22]). At $O(p^4)$, there are two possibilities: either one-loop graphs composed only of lowest-order vertices ($V_{n>2} = 0$, $L = 1$), or tree graphs with exactly one insertion from $\mathcal{L}_4$ ($V_4 = 1$, $V_{n>4} = 0$, $L = 0$) (Fig.1 (b)). Finally, at $O(p^6)$, Eq.(26) allows

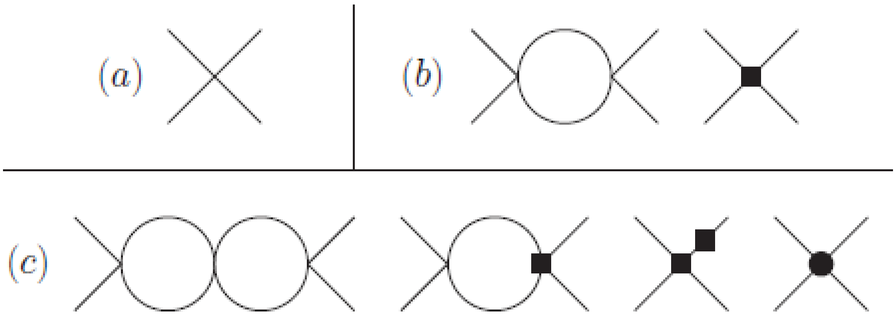

Figure 1: Feynman graphs contributing to $\pi\pi$ scattering at (a) $O(p^2)$, (b) $O(p^4)$, (c) $O(p^6)$. The square denotes vertices from $\mathcal{L}_4$, the circle a vertex from $\mathcal{L}_6$.

for four different types of graphs: two-loop graphs with $\mathcal{L}_2$ vertices ($V_{n>2} = 0$, $L = 2$); one-loop graphs with one vertex from $\mathcal{L}_4$ ($V_4 = 1$, $V_{n>4} = 0$, $L = 1$); tree graphs with two insertions from $\mathcal{L}_4$ ($V_4 = 2$, $V_{n>4} = 0$, $L = 0$); and tree graphs with one insertion from $\mathcal{L}_6$ ($V_4 = 0$, $V_6 = 1$, $V_{d>6} = 0$, $L = 0$),(Fig.1 (c)).

Calculating loop graphs, we might expect, that a given amplitude is proportional to some power of the parameter $p/\Lambda_\chi$, where $1/\Lambda_\chi$ plays a role of expansion parameter of the effective Lagrangian. There is an estimate of $\Lambda_\chi$ based on loop expansion [23]:

$$\Lambda_\chi \sim 4\pi F_0 \approx 1.2 \text{ GeV}, \tag{27}$$

as well as improved estimate [24, 25]:

$$\Lambda_\chi \sim \frac{4\pi F_0}{\sqrt{N_f}},$$

where $N_f$ is the number of light flavors ($N_f$=2, 3). The former estimate stems from the fact, that the greater $N_f$ is, the more number of mesons can run in loops. Therefore, one would expect considerably better convergence of the chiral expansion in the $\text{SU}(2)_L \times \text{SU}(2)_R$ framework, because in this case $N_f = 2$ and $|p| = O(M_\pi)$.

In addition, note that effective theory contains Goldstone bosons as the only dynamical degrees of freedom. Therefore, it must fail once the energy reaches the resonance region, hence for $p^2/\Lambda_\chi^2 \approx p^2/M_{res}^2 \approx 1$. The lightest resonance, observed in $\pi\pi$ scattering in the $I = l = 1$ channel is a $\rho$ resonance: $M_{res} = M_\rho = 770$MeV. It is therefore appropriate to choose

$$\Lambda_\chi \sim M_\rho \approx 770\text{MeV}, \tag{28}$$

which is consistent with the estimate Eq.(27).

Now, based on general formalism, outlined in Sec.2.2 one can construct, in term of building blocks Eq.(16), the effective Lagrangian at higher orders. However, the number of independent terms and corresponding low-energy constants increases rapidly at higher orders. Note that in contrast to $\mathcal{L}_2$, which has the same form for both SU(2) and SU(3), the number of terms at higher orders is different in both theories because, although both have the same most general SU($N$) Lagrangian, certain matrix-trace (Cayley-Hamilton) relations render some of the structures redundant, such that the minimal numbers of independent terms differ.One can summarize for chiral SU($N_f$), $N_f = (2,3)$ as follows [22]

$$\begin{array}{lllll} O(p^2)\text{: } \mathcal{L}_2 & \text{contains} & (2,2) & \text{constants} & (F_0,\, B_0), \\ O(p^4)\text{: } \mathcal{L}_4 & \text{contains} & (7,10) & \text{constants} & [13,\, 14], \\ O(p^6)\text{: } \mathcal{L}_6 & \text{contains} & (53,90) & \text{constants} & [26] \end{array}$$

(discounting so-called contact terms that depend on external fields only).

For the two-flavor case the effective Lagrangian at next-to-leading order has the form

$$\begin{aligned} \mathcal{L}_4 = \mathcal{L}_{p^4} \;\; = \;\; & \frac{l_1}{4} \langle\, d^\mu U^+ d_\mu U \,\rangle^2 \;+\; \frac{l_2}{4} \langle\, d^\mu U^+ d^\nu U \,\rangle \langle\, d_\mu U^+ d_\nu U \,\rangle \\ & + \frac{l_3}{16} \langle\, \chi^+ U + U^+ \chi \,\rangle^2 \;+\; \frac{l_4}{4} \langle\, d^\mu U^+ d_\mu \chi + d^\mu \chi^+ d_\mu U \,\rangle \\ & + l_5 \langle\, f^{\mathrm{R}}_{\mu\nu} U f^{\mathrm{L}\,\mu\nu} U^+ \,\rangle \;+\; \frac{i l_6}{2} \langle\, f^{\mathrm{R}}_{\mu\nu} d^\mu U d^\nu U^+ + f^{\mathrm{L}}_{\mu\nu} d^\mu U^+ d^\nu U \,\rangle \\ & - \frac{l_7}{16} \langle\, \chi^+ U - U^+ \chi \,\rangle^2 \;+\; \frac{1}{4} (h_1 + h_3) \langle\, \chi^+ \chi \,\rangle \\ & + \frac{1}{2} (h_1 - h_3)\, Re(\det \chi) \;-\; h_2 \langle\, f^{\mathrm{R}}_{\mu\nu} f^{\mathrm{R}\,\mu\nu} + f^{\mathrm{L}}_{\mu\nu} f^{\mathrm{L}\,\mu\nu} \,\rangle \;. \end{aligned} \tag{29}$$

and satisfies local chiral invariance, Lorentz invariance, P and C [13, 27, 28].The symbol $\langle \cdots \rangle$ denotes the trace in flavor space. The low-energy behavior of the Green functions at next-to-leading order is determined by 7 low-energy coupling constants (chiral couplings) $l_1, \cdots, l_7$. The terms proportional to $h_1, h_2, h_3$ do not contain the pseudoscalar fields and therefore not directly measurable. Although, in principle, chiral couplings are calculable functions of $\Lambda_{QCD}$ and the heavy quark masses, the main source of information about these couplings is low-energy phenomenology.

As one can see, the Lagrangian $\mathcal{L}_4$ contains terms which are not presented in $\mathcal{L}_2$. This is the general feature of effective field theories, which are non-renormalizable (i.e. an infinite number of counterterms is required). However, order by order in the momentum expansion they define a renormalizable theory. If we use a regularization which preserves the symmetries of the Lagrangian , such ads dimensional regularization ,the counter-terms needed to renormalize the theory will be necessarily symmetric. Since the ChPT Lagrangian $\mathcal{L}_{\text{eff}}$ is the most general chiral invariant Lagrangian,i.e it contains all terms permitted by the symmetry, the divergences can then be absorbed in a renormalization of the coupling constants occurring in the Lagrangian[29].At one loop , the ChPT divergences are $O(p^4)$ and can be eliminated by an appropriate renormalization of the low-energy constants $l_i$ and $h_i$.

# 4 Virtual photons in ChPT for mesons

## 4.1 Effective Lagrangian at $O(p^2)$

We have consider in previous section ChPT for mesons in the strong sector. However,since there are electrically charged mesons ($\pi^+, \pi^-, K^+$, see Eq.(14)) it is necessary to include electromagnetic interactions in ChPT framework to analyze the electromagnetic corrections to meson masses, scattering amplitudes an so on. In a first step, the electromagnetic field is made dynamical by including the appropriate kinetic term and by enlarging the external vector field in the generating functional

$$v_\mu \to v_\mu - QA_\mu, \tag{30}$$

where $A_\mu$ is photon field and $Q$ is a quark charge matrix, which is given in a two-flavor case by

$$Q = \frac{e}{3}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle Q \rangle^2 = \frac{1}{5}\langle Q^2 \rangle$$

Such an inclusion of electromagnetism via minimal substitution does not generate the most general effects due to *virtual* photons.

Let us consider the part of the QCD Lagrangian coupling quarks to photons, decomposed into chiral components

$$\mathcal{L}_{\text{em}} = -\bar{q}QA_\mu\gamma^\mu q = -\bar{q}_RQA_\mu\gamma^\mu q_R - \bar{q}_LQA_\mu\gamma^\mu q_L,$$

If we introduce the so called spurion fields $Q_R(x)$, $Q_L(x)$ and rewrite $\mathcal{L}_{\text{em}}$ as follows

$$\mathcal{L}_{\text{em}} = -\bar{q}_RQ_RA_\mu\gamma^\mu q_R - \bar{q}_LQ_LA_\mu\gamma^\mu q_L,$$

then $\mathcal{L}_{\text{em}}$ will be locally chiral invariant, if the spurions transform under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ as

$$Q_I \to g_I\, Q_I\, g_I^\dagger\,, \quad g_I \in SU(2)_I\,, \quad I = L, R\;. \tag{31}$$

In the presence of electromagnetism a consistent expansion scheme is obtain if the electric charge $e$ is of chiral order $O(p)$ [31]

$$e,\, Q_R,\, Q_L = O(p), \quad A_\mu = O(1).$$

Using the spurions $Q_R(x)$, $Q_L(x)$ as additional building blocks and the counting rule for them, one can construct the most general Lagrangian, which includes electromagnetic interactions and which is consistent with the chiral symmetry, $P$ and $C$ invariance. One then sets the spurion fields to the constant charge matrix Q:

$$Q_R(x) = Q_L(x) = Q$$

At leading order for an arbitrary number of light quark flavors the effective Lagrangian has a form [30, 31]

$$\begin{aligned} \mathcal{L}_2^{(Q)} &= \frac{F_0^2}{4}\langle\, d^\mu U^+ d_\mu U \,+\, \chi^+ U + U^+\chi \,\rangle \\ &\quad -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \,-\, \frac{1}{2a}\,(\partial^\mu A_\mu)^2 \,+\, C\,\langle\, Q_R U Q_L U^+ \,\rangle\ . \end{aligned} \tag{32}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the the photon field strength tensor, $a$ the gauge fixing parameter, $d_\mu$ the generalized covariant derivative,

$$d_\mu U \;=\; \partial_\mu U - i(v_\mu + Q_R A_\mu + a_\mu)U + iU(v_\mu + Q_L A_\mu - a_\mu) \tag{33}$$

The term in $\mathcal{L}_2^{(Q)}$, proportional to coupling constant $C$ gives an electromagnetic contribution to the masses of the charged mesons:

$$\left(M_{\pi^+}^2 - M_{\pi^0}^2\right)_{\rm em} = \left(M_{K^+}^2 - M_{K^0}^2\right)_{\rm em} = \frac{2Ce^2}{F_0^2}\ . \tag{34}$$

The equality of electromagnetic contributions to pion and kaon mass differences in the chiral limit is known as Dashen's theorem [32].

From $\mathcal{L}_2^{(Q)}$ one can derive equation of motion for matrix $U$

$$\begin{aligned} & d_\mu d^\mu \bar U \bar U^+ - \bar U d_\mu d^\mu \bar U^+ + \bar U\chi^+ - \chi\bar U^+ - \frac{1}{2}\langle \bar U\chi^+ - \chi\bar U^+\rangle \\ & \qquad + \frac{4C}{F_0^2}\left(\bar U Q \bar U^+ Q - Q\bar U Q \bar U^+\right) \;=\; 0, \end{aligned} \tag{35}$$

and for the photon field $A_\mu$

$$\left[g_{\mu\nu}\Box - \left(1 - \frac{1}{a}\right)\partial_\mu\partial_\nu\right]\bar A^\nu + \frac{iF_0^2}{2}\langle d_\mu \bar U[\bar U^+, Q]\rangle = 0. \tag{36}$$

## 4.2 Effective Lagrangian at $O(p^4)$

The Lagrangian $\mathcal{L}_2^{(Q)}$ generates one-loop graphs consisting of meson and photon lines. They are of order $O(p^4)$ and contain divergences, which should be absorbed by adding tree graphs, evaluated with the next-to-leading order Lagrangian $\mathcal{L}_4^{(Q)}$. Consider loop expansion from point of view of path integral formulation of quantum filed theory. The generating functional reads, up to and including terms of order $O(p^4)$

$$e^{iZ[v,a,s,p]} = \int [dU][dA_\mu] e^{i\int d^4x \left\{\mathcal{L}_2^{(Q)} + \mathcal{L}_4^{(Q)}\right\}}, \tag{37}$$

where $[dA_\mu]$ means the path integral measure for electromagnetic field. One should calculate $Z[v,a,s,p]$ at one-loop level. To this purpose, we note that the classical field theory associated with a given Lagrangian is equivalent to the set of tree graphs of the corresponding quantum field theory. Thus if we use the classical field equations to evaluate $Z[v,a,s,p]$, then $Z[v,a,s,p]$ generates Green functions at tree approximation (leading order) [33, 34].

Since the vertices of the Lagrangian $\mathcal{L}_4^{(Q)}$ only occur in tree graphs, the contribution from $\mathcal{L}_4^{(Q)}$ to the generating functional can be calculated by evaluating the action $\int dx\, \mathcal{L}_4^{(Q)}$ at the classical solution of the equations of motion. Therefore the most general Lagrangian at $O(p^4)$ can be simplified with the help of the equations of motion.

The next-to-leading order Lagrangian in the presence of virtual photons was constructed in Ref. [31]. It has a following form for two-flavor case[27, 35]:

$$\begin{aligned}
\bar{\mathcal{L}}_4^{(Q)} \;=\;& \bar{\mathcal{L}}_{p^4} + F_0^2\{k_1\langle d^\mu U^+ d_\mu U\rangle\langle Q^2\rangle + k_2\langle d^\mu U^+ d_\mu U\rangle\langle QUQU^+\rangle \\
&+k_3(\langle d^\mu U^+ QU\rangle\langle d_\mu U^+ QU\rangle + \langle d^\mu U QU^+\rangle\langle d_\mu U QU^+\rangle) \\
&+k_4\langle d^\mu U^+ QU\rangle\langle d_\mu U QU^+\rangle + k_5\langle \chi^+ U + U^+\chi\rangle\langle Q^2\rangle \\
&+k_6\langle \chi^+ U + U^+\chi\rangle\langle QUQU^+\rangle \\
&+k_7\langle(\chi U^+ + U\chi^+)Q + (\chi^+ U + U^+\chi)Q\rangle\langle Q\rangle \\
&+k_8\langle(\chi U^+ - U\chi^+)QUQU^+ + (\chi^+ U - U^+\chi)QU^+QU\rangle \\
&+k_9\langle d_\mu U^+[c_{\rm R}^\mu Q, Q]U + d_\mu U[c_{\rm L}^\mu Q, Q]U^+\rangle \\
&+k_{10}\langle c_{\rm R}^\mu Q U c_{{\rm L}\mu} Q U^+\rangle + k_{11}\langle c_{\rm R}^\mu Q c_{{\rm R}\mu} Q + c_{\rm L}^\mu Q c_{{\rm L}\mu} Q\rangle\} \\
&+F_0^2\{k_{12}\langle QUQU^+\rangle^2 + k_{13}\langle QUQU^+\rangle\langle Q^2\rangle + k_{14}\langle Q^2\rangle^2\},
\end{aligned} \tag{38}$$

where it is assumed that $U = \bar{U}, A_\mu = \bar{A}_\mu$ are classical solutions of equations of motion. The covariant derivatives $c_R^\mu Q_R$, $c_L^\mu Q_L$ are defined as

$$c_I^\mu Q_I = \partial_\mu Q_I - i[I_\mu, Q_I], \qquad\qquad I = R, L\,.$$

They transform under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ in the same way as $Q_R$ and $Q_L$

$$c_I^\mu Q_I \to g_I\,(c_I^\mu Q_I)\,g_I^\dagger\,, \quad g_I \in SU(2)_I\,, \quad I = L, R\,. \tag{39}$$

Since at later stage, one sets $Q_R = Q_L = Q = \text{const}$, then $c_\mu^I Q = -i[I_\mu, Q]$.

### 4.3 One-loop generating functional

The generating functional of Eq. (37) becomes

$$e^{iZ[v,a,s,p]} = e^{i\int d^4x\bar{\mathcal{L}}_4^{(Q)}} \int [dU][dA_\mu] e^{i\int d^4x\mathcal{L}_2^{(Q)}}$$

To evaluate the one-loop graphs produced by the Lagrangian $\mathcal{L}_2^{(Q)}$, we expand the fields $U(x), A_\mu(x)$ in the neighborhood of the classical solutions $\bar{U}$, $\bar{A}_\mu$ [36]:

$$\begin{aligned} U &= ue^{i\xi/F_0}u = u\left(\mathbf{1} + i\frac{\xi}{F_0} - \frac{1}{2}\frac{\xi^2}{F_0^2} + \cdots\right)u \\ &= \bar{U} + \frac{i}{F_0}u\xi zu - \frac{1}{2F_0^2}u\xi^2 u + \ldots \\ A_\mu &= \bar{A}_\mu + \epsilon_\mu, \end{aligned} \tag{40}$$

where $\bar{U} = u^2$ and $\xi$ is a traceless hermitian matrix, $\xi = \sum_a \xi^a\tau^a$, and $\tau^a$ denote the Pauli matrices. Then we substitute this expansion in the action $S = \int dx\mathcal{L}_2^{(Q)}$ and keep only terms, quadratic in the fluctuations $\xi$, $\epsilon_\mu$. As a result we obtain [31, 27]

$$S = \int dx\bar{\mathcal{L}}_2^{(Q)} - \frac{1}{2}\int dx\eta_A D^{AB}\eta_B,$$

where the fluctuations are collected in a new flavor space elements $\eta_A = (\xi_a, \epsilon_\mu) = z(\xi_1, \ldots, \xi_3, \epsilon_0, \ldots, \epsilon_3)$ and matrix $D$ is the differential operator defined as follows:

$$D = D_0 + \delta, \tag{41}$$

$$D_0 = \begin{pmatrix} \partial^2\delta^{ab} & 0 \\ 0 & -\partial^2 g^{\sigma\rho} + \left(1 - \frac{1}{a}\right)\partial^\sigma\partial^\rho \end{pmatrix}, \tag{42}$$

$$\delta(x) = \{Y_\mu, \partial^\mu\} + Y_\mu Y^\mu + \Lambda, \tag{43}$$

with

$$Y_\mu(x) = \begin{pmatrix} \Gamma_\mu^{ab} & X_\mu^{a\rho} \\ X_\mu^{\sigma b} & 0 \end{pmatrix}, \quad \Lambda(x) = \begin{pmatrix} \sigma^{ab} & -\frac{1}{2}\gamma^{a\rho} \\ -\frac{1}{2}\gamma^{\sigma b} & -\rho g^{\sigma\rho} \end{pmatrix}. \tag{44}$$

The elements of these matrices are given by the expressions:

$$\begin{aligned} \Gamma_\mu^{ab} &= -\frac{1}{2}\langle[\tau^a, \tau^b]\Gamma_\mu\rangle, \\ X_\mu^{a\rho} &= -X_\mu^{\rho a} = X^a\delta_\mu^\rho, \quad X^a = -\frac{1}{4}\langle H_L\tau^a\rangle, \\ \sigma^{ab} &= \frac{1}{2}\langle[\Delta_\mu, \tau^a][\Delta^\mu, \tau^b]\rangle + \frac{1}{4}\langle\{\tau^a, \tau^b\}\sigma\rangle - \frac{F_0^2}{4}\langle H_L\tau^a\rangle\langle H_L\tau^b\rangle \end{aligned}$$

$$
\begin{aligned}
& -\frac{C}{8F_0^2}\left\{\langle[H_R+H_L,\tau^a][H_R-H_L,\tau^b]+a\leftrightarrow b\rangle\right\}, \\
\gamma^{a\rho} &= \gamma^{\rho a}=F_0\langle\Big([H_R,\Delta^\rho]+\frac{1}{2}D^\rho H_L\Big)\tau^a\rangle, \\
\rho &= \frac{3}{8}F_0^2\langle H_L^2\rangle, \qquad (45)
\end{aligned}
$$

where

$$
\begin{aligned}
D_\mu H_L &= \partial_\mu H_L+[\Gamma_\mu,H_L] \\
\Gamma_\mu &= \frac{1}{2}[u^+,\partial_\mu u]-\frac{1}{2}iu^+\bar{R}_\mu u-\frac{1}{2}u\bar{L}_\mu u^+, \\
\Delta_\mu &= \frac{1}{2}u^+d_\mu\bar{U}u^+=-\frac{1}{2}ud_\mu\bar{U}^+u, \\
H_R &= u^+Q_Ru+uQ_Lu^+, \\
H_L &= u^+Q_Ru-uQ_Lu^+, \\
\sigma &= \frac{1}{2}(u^+\chi u^++u\chi^+u). \qquad (46)
\end{aligned}
$$

The generating functional thus takes the form

$$
e^{iZ[v,a,s,p]}=e^{i\int dx\left\{\bar{\mathcal{L}}_2^{(Q)}+\bar{\mathcal{L}}_4^{(Q)}\right\}}\int[d\xi_a][d\epsilon_\mu]e^{-\frac{i}{2}\int dx\eta_A D^{AB}\eta_B}.
$$

The remaining path integral over fluctuations reduces to a Gaussian integral and we finally obtain $Z[v,a,s,p]$ at order $O(p^4)$:

$$
Z[v,a,s,p]=\int dx\bar{\mathcal{L}}_2^{(Q)}+\int dx\bar{\mathcal{L}}_4^{(Q)}+\frac{i}{2}\ln\det D, \qquad (47)
$$

where all quantities are to be evaluated at the classical solutions $\bar{U}(x),\bar{A}_\mu(x)$. The determinant of the operator $D$ requires renormalization, since it contains divergences of one-loop graphs with arbitrary number of external legs. These divergences may be absorbed by an appropriate renormalization of the low-energy coupling constants in the Lagrangian $\bar{\mathcal{L}}_4^{(Q)}$ of Eq. (29):

$$
\begin{aligned}
l_i &= l_i^r(\mu)+\gamma_i\lambda, \\
h_i &= h_i^r(\mu)+\delta_i\lambda, \\
k_i &= k_i^r(\mu)+\sigma_i\lambda, \qquad (48)
\end{aligned}
$$

where $\lambda$ is defined as

$$
\lambda=\frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4}-\frac{1}{2}[\ln(4\pi)+\Gamma'(1)+1]\right\}
$$

with $d$ denoting the number of space-time dimensions. The renormalized constants $l_i^r(\mu), h_i^r(\mu)$, $k_i^r(\mu)$ are finite and depend on the scale $\mu$ introduced by dimensional regularization. The coefficients $\gamma_i$, $\delta_i$, $\sigma_i$ are some numbers, which has to be chosen in such a way, that the generating functional (47) is finite. The resulting $Z[v,a,s,p]$ generates the general solution of the Ward identities at next-to-leading order.

We see that in order to determine the coefficients $\gamma_i$, $\delta_i$, $\sigma_i$ we need to regularize the determinant of the operator $D$. Thus, we have to separate out the divergent part of the one-loop generating functional

$$Z_{\text{one loop}} = \frac{i}{2} \ln \det D.$$

There exists the so-called heat kernel method [37], which allows to calculate the divergent part of the $\ln \det D$. However, this method can be applied (at least, without modifications) only to the differential operators of so-called minimal kind. The operator $D$ is nonminimal in general. It becomes minimal when the gauge parameter is set to 1: $a = 1$ (Feynman gauge). This is the case considered in Ref. [31]. Using the heat kernel method for the operator $D$, one obtains the divergent part of the one-loop functional [13, 31]:

$$Z^{a=1}_{\text{one loop}} = -\frac{1}{16\pi^2}\frac{1}{d-4}\int d^4x \; \mathrm{Sp}\left(\frac{1}{12}Y_{\mu\nu}Y^{\mu\nu} + \frac{1}{2}\Lambda^2\right) + \text{finite parts}, \tag{49}$$

where Sp means the trace in the flavor space $\eta^A$ and $Y_{\mu\nu}$ denotes the field strength tensor of $Y_\mu$,

$$Y_{\mu\nu} = \partial_\mu Y_\nu - \partial_\nu Y_\mu + [Y_\mu, Y_\nu].$$

One then can find the coefficients $\gamma_i$, $\delta_i$, $\sigma_i$. The coefficients $\gamma_i$, $\delta_i$ are specified in Ref. [13], and $\sigma_i$ in Ref. [27].

## 4.4 $\beta$-functions in arbitrary gauge

Our goal from now on, as well as the main aim of the present thesis, is to **find the $\sigma_i$ in an arbitrary covariant gauge** $a$ .

As we have already mentioned, the coefficients $\sigma_i$, or alternatively the $\beta$-functions, defined from Eq. (48) as

$$\beta_i = \mu \frac{dk_i^r(\mu)}{d\mu} = -\frac{1}{16\pi^2}\sigma_i,$$

were calculated in Feynman gauge $a = 1$. In order to extend the evaluation of $\sigma_i$ to the case of the arbitrary covariant gauge , we chose another method of calculation of the divergent part of the one-loop functional $Z_{\text{one loop}}$ [13, 38]. In the first step, we expand the determinant of $D$ of Eq. (41) in powers of the operator $\delta$ :

$$\begin{aligned} Z_{\text{one loop}} &= \frac{i}{2}\ln\det(D_0+\delta) = \frac{i}{2}\ln\det D_0 + \frac{i}{2}\text{Tr}(D_0^{-1}\delta) \\ &-\frac{i}{4}\text{Tr}(D_0^{-1}\delta D_0^{-1}\delta) + \frac{i}{6}\text{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) \\ &-\frac{i}{8}\text{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) + \text{finite parts}, \end{aligned} \tag{50}$$

where trace Tr denotes, in coordinate space, the integral $\text{Tr}\{\ldots\} = \int \text{dx}\langle \text{x}|\text{Sp}\{\ldots\}|\text{x}\rangle$. We have written out only terms, which contain the ultraviolet divergences. In momentum space at large momenta the matrix element of the operator $D_0$ is proportional to $1/k^2$, while the matrix element of the operator $\delta$ is proportional to $k$. Each trace in the sum at large momenta is proportional to the integral $\int d^4k \frac{1}{k^n}$, which is divergent only for $n \leq 4$. Therefore, divergent are only traces presented in Eq. (50). We checked that in case of the minimal operator $D$, the expansion (50) leads to the same divergent part of Eq. (49), obtained by the heat kernel method (this was done in the strong sector, without virtual photons). Below we will use dimensional regularization as a convenient one.

To perform the calculations in the arbitrary gauge we at first explicitly expand the traces in the flavor space $\eta^A$. For the first trace we have:

$$\text{Sp}\{D_0^{-1}\delta\} = (D_0^{-1}\delta)_A^A = (D_0^{-1})_{ab}(\delta)^{ba} + (D_0^{-1})_{\sigma\rho}(\delta)^{\rho\sigma},$$

where we used the fact that $(D_0^{-1})_{a\rho} = (D_0^{-1})_{\sigma b} = 0$. Inserting necessary number of completeness relation in coordinate space $\int dx|x\rangle\langle x| = 1$, we obtain

$$\text{Tr}(D_0^{-1}\delta) = \int dxdy\, \left\{ \langle x|(D_0^{-1})_{ab}|y\rangle\langle y|(\delta)^{ba}|x\rangle + \langle x|(D_0^{-1})_{\sigma\rho}|y\rangle\langle y|(\delta)^{\rho\sigma}|x\rangle \right\}.$$

The matrix elements of the operators $D_0^{-1}$ and $\delta$ have the following form:

$$\langle x|(D_0^{-1})^{ab}|y\rangle = \delta^{ab}\Delta(x-y),$$

$$\begin{aligned}\langle x|(D_0^{-1})^{\mu\nu}|y\rangle &= -g^{\mu\nu}\Delta(x-y)+\Delta^{\mu\nu}(x-y),\\ \langle y|(\delta)^{AB}|x\rangle &= 2Y_\mu^{AB}(y)\partial_y^\mu\delta(x-y)+c^{AB}(y)\delta(x-y),\end{aligned} \tag{51}$$

where

$$\begin{aligned}\Delta(x-y) &= \int\frac{d^dk}{(2\pi)^d}\frac{e^{-ik(x-y)}}{-k^2},\\ \Delta^{\mu\nu}(x-y) &= (a-1)\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{k^4}e^{-ik(x-y)},\\ c(x) &= (\partial_\mu Y^\mu)+Y_\mu Y^\mu+\Lambda.\end{aligned} \tag{52}$$

Using Eq. (51), we separate the divergent terms, that depend on the gauge parameter (through $\Delta_{\mu\nu}(x)$), and hereby obtain for $\mathrm{Tr}(D_0^{-1}\delta)$:

$$\mathrm{Tr}(D_0^{-1}\delta)=\mathrm{Tr}(D_0^{-1}\delta)^{a=1}+\int dxdy\Delta_{\sigma\rho}(x-y)\langle y|(\delta)^{\rho\sigma}|x\rangle,$$

where $\mathrm{Tr}(D_0^{-1}\delta)^{a=1}$ means the trace, which is calculated in Feynman gauge. We note, that since $Y_\mu^{\sigma\rho}(x)=0$, then the matrix element of the operator $\delta$ simplifies to

$$\langle y|(\delta)^{\sigma\rho}|x\rangle=c^{\sigma\rho}(y)\delta(x-y). \tag{53}$$

This fact considerably reduces the number of divergent integrals, that one has to evaluate. Substituting the expression (53), partially integrating over the coordinate $y$ and then taking integral over $x$, we get

$$\mathrm{Tr}(D_0^{-1}\delta)=\mathrm{Tr}(D_0^{-1}\delta)^{a=1}+\int dy\,\Delta_{\sigma\rho}(0)c^{\rho\sigma}(y).$$

The quantity $\Delta_{\sigma\rho}(0)$, which is the integral in momentum space, is zero in dimensional regularization,

$$\int\frac{d^dk}{(2\pi)^d(k^2)^m}=0,\quad \text{for any } m;\quad \Delta_{\sigma\rho}(0)=(a-1)\int\frac{d^dk}{(2\pi)^d}\frac{k_\sigma k_\rho}{k^4}\sim g_{\sigma\rho}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}=0.$$

Thus, $\mathrm{Tr}(D_0^{-1}\delta)=\mathrm{Tr}(D_0^{-1}\delta)^{a=1}$. We perform the same steps for other traces in the expansion (50). Details are provided in appendix B. The divergent part of the one-loop functional in the arbitrary gauge is given by the expression

$$\begin{aligned}divZ_{\text{one loop}} &= divZ_{\text{one loop}}^{a=1}+\frac{1}{4}F_0^2(1-a)\frac{1}{16\pi^2\epsilon}\int dx\Big\{\langle[H_L,\Delta_\mu]^2\rangle-\langle[H_R,\Delta_\mu]^2\rangle\\ &+\langle H_L^2\sigma\rangle-2\langle[H_R,\Delta_\mu]G^\mu\rangle-\frac{3}{4}\langle G_\mu G^\mu\rangle-\frac{1}{8}F_0^2Z\langle[H_R+H_L,H_R-H_L]^2\rangle\Big\},\end{aligned} \tag{54}$$

where $div$ means divergent part, $Z = C/F_0^4$ and $G^\mu$ is defined as

$$G^\mu = u^+ c_R^\mu Q u - u c_L^\mu Q u^+.$$

Next, with help of Eq. (46), we simplify the result:

$$\begin{aligned}
&\langle [H_L, \Delta_\mu]^2\rangle - \langle [H_R, \Delta_\mu]^2\rangle = -\langle d^\mu \bar{U}^+ d_\mu \bar{U} Q \bar{U}^+ Q \bar{U} + d^\mu \bar{U} d_\mu \bar{U}^+ Q \bar{U} Q \bar{U}^+\rangle + 2\langle Q d_\mu \bar{U} Q d^\mu \bar{U}^+\rangle,\\
&\langle H_L^2 \sigma\rangle = \frac{1}{2}\langle (\chi\bar{U}^+ + \bar{U}\chi^+ + \chi^+\bar{U} + \bar{U}^+\chi) Q^2\rangle\\
&\qquad -\frac{1}{2}\langle (\chi\bar{U}^+ + \bar{U}\chi^+) Q\bar{U}Q\bar{U}^+ + (\chi^+\bar{U} + \bar{U}^+\chi) Q\bar{U}^+ Q\bar{U}\rangle,\\
&\langle [H_R, \Delta_\mu] G^\mu\rangle = -\frac{1}{2}\langle d_\mu \bar{U}^+ [c_R^\mu Q, Q]\bar{U} + d_\mu \bar{U}[c_L^\mu Q, Q]\bar{U}^+\rangle\\
&-\frac{1}{2}\langle d_\mu \bar{U}^+ c_R^\mu Q \bar{U} Q + d_\mu \bar{U} Q \bar{U}^+ c_R^\mu Q + d_\mu \bar{U} c_L^\mu Q \bar{U}^+ Q + d_\mu \bar{U}^+ Q \bar{U} c_L^\mu Q\rangle,\\
&\langle G_\mu G^\mu\rangle = \langle c_R^\mu Q c_{R\mu} Q + c_L^\mu Q c_{L\mu} Q\rangle - 2\langle c_R^\mu Q \bar{U} c_{L\mu} Q \bar{U}^+\rangle,\\
&\langle [H_R + H_L, H_R - H_L]^2\rangle = 32\langle Q\bar{U}Q\bar{U}^+Q\bar{U}Q\bar{U}^+ - Q^2\bar{U}Q^2\bar{U}^+\rangle.
\end{aligned} \tag{55}$$

The second trace in the expression for $\langle [H_R, \Delta_\mu] G^\mu\rangle$ can be transformed, using partial integration and the equation of motion, obeyed by $\bar{U}$ [39]. We obtain

$$\begin{aligned}
&\langle d_\mu \bar{U}^+ c_R^\mu Q \bar{U} Q + d_\mu \bar{U} Q \bar{U}^+ c_R^\mu Q + d_\mu \bar{U} c_L^\mu Q \bar{U}^+ Q + d_\mu \bar{U}^+ Q \bar{U} c_L^\mu Q\rangle =\\
&\langle d^\mu \bar{U}^+ d_\mu \bar{U} Q \bar{U}^+ Q \bar{U} + d^\mu \bar{U} d_\mu \bar{U}^+ Q \bar{U} Q \bar{U}^+\rangle - 2\langle Q d_\mu \bar{U} Q d^\mu \bar{U}^+\rangle\\
&+\frac{1}{2}\langle (\chi\bar{U}^+ - \bar{U}\chi^+) Q\bar{U}Q\bar{U}^+ + (\chi^+\bar{U} - \bar{U}^+\chi) Q\bar{U}^+ Q\bar{U}\rangle\\
&+4F_0^2 Z\langle Q\bar{U}Q\bar{U}^+Q\bar{U}Q\bar{U}^+ - Q^2\bar{U}Q^2\bar{U}^+\rangle.
\end{aligned} \tag{56}$$

Thus, the final result for $divZ_{\text{one loop}}$ is

$$\begin{aligned}
&divZ_{\text{one loop}} = divZ^{a=1}_{\text{one loop}} - \frac{1}{16\pi^2(d-4)}(1-a)F_0^2 \int dx \Big\{\\
&\frac{1}{4}\langle (\chi\bar{U}^+ + \bar{U}\chi^+ + \chi^+\bar{U} + \bar{U}^+\chi) Q^2\rangle\\
&-\frac{1}{4}\langle (\chi\bar{U}^+ + \bar{U}\chi^+) Q\bar{U}Q\bar{U}^+ + (\chi^+\bar{U} + \bar{U}^+\chi) Q\bar{U}^+ Q\bar{U}\rangle\\
&+\frac{1}{4}\langle (\chi\bar{U}^+ - \bar{U}\chi^+) Q\bar{U}Q\bar{U}^+ + (\chi^+\bar{U} - \bar{U}^+\chi) Q\bar{U}^+ Q\bar{U}\rangle\\
&+\frac{1}{2}\langle d_\mu \bar{U}^+ [c_R^\mu Q, Q]\bar{U} + d_\mu \bar{U}[c_L^\mu Q, Q]\bar{U}^+\rangle + \frac{3}{4}\langle c_R^\mu Q \bar{U} c_{L\mu} Q \bar{U}^+\rangle\\
&-\frac{3}{8}\langle c_R^\mu Q c_{R\mu} Q + c_L^\mu Q c_{L\mu} Q\rangle\Big\}.
\end{aligned} \tag{57}$$

In case of two flavors the first term of the integrand in Eq.(57) can be transformed, using the following trace identity for $2 \times 2$ matrices $A$, $B$ and $C$ :

$$\langle ABC\rangle + \langle BAC\rangle - \langle A\rangle\langle BC\rangle - \langle B\rangle\langle AC\rangle - \langle C\rangle\langle AB\rangle + \langle A\rangle\langle B\rangle\langle C\rangle = 0$$

$$\langle(\chi\bar{U}^+ + \bar{U}\chi^+ + \chi^+\bar{U} + \bar{U}^+\chi)Q^2\rangle = \langle\chi^+\bar{U} + \bar{U}^+\chi\rangle\langle Q^2\rangle + \langle\chi^+\bar{U} + \bar{U}^+\chi\rangle\langle Q\bar{U}Q\bar{U}^+\rangle$$

Finally, if we write the coefficients $\sigma_i$ as

$$\sigma_i = \sigma_i^{a=1} + \sigma_i^a,$$

where $\sigma_i^{a=1}$ are ones calculated in Feynman gauge, then $\sigma_i^a$ can be directly read off from Eq. (57). They are presented in Table 2.

| $i$ | $\sigma_i^a$ | $\sigma_i$ |
|---|---|---|
| 1 | 0 | $-\frac{27}{20} - \frac{1}{5}Z$ |
| 2 | 0 | $2Z$ |
| 3 | 0 | $-\frac{3}{4}$ |
| 4 | 0 | $2Z$ |
| 5 | $\frac{1}{4}(1-a)$ | $-\frac{1}{5}Z - \frac{1}{4}a$ |
| 6 | $-\frac{1}{4}(1-a)$ | $2Z + \frac{1}{4}a$ |
| 7 | 0 | 0 |
| 8 | $\frac{1}{4}(1-a)$ | $\frac{3}{8} - Z - \frac{1}{4}a$ |
| 9 | $\frac{1}{2}(1-a)$ | $\frac{3}{4} - \frac{1}{2}a$ |
| 10 | $\frac{3}{4}(1-a)$ | $\frac{3}{4}(1-a)$ |
| 11 | $-\frac{3}{8}(1-a)$ | $-\frac{3}{8}(1-a)$ |
| 12 | 0 | $\frac{3}{2} - \frac{12}{5}Z + \frac{84}{25}Z^2$ |
| 13 | 0 | $-3 - \frac{3}{5}Z - \frac{12}{5}Z^2$ |
| 14 | 0 | $\frac{3}{2} + 3Z + 12Z^2$ |

Table 2: *The coefficients $\sigma_i$ and their gauge dependent parts $\sigma_i^a$.*

## 4.5 Discussion of the result

Let us now discuss the validity of the result. Fist of all, gauge-dependence of the $\sigma_9$ is an agreement with that only one calculated in Ref.[40].

We checked,further, that the parts of the $\sigma_i^a$ do not introduce the dependence of the physical quantities on the renormalization scale $\mu$, as it should. Namely, we considered the masses of pions and $\pi\pi$ scattering amplitudes, calculated at one-loop level with virtual photons included [27, 35]. The expressions for them contain certain combinations of coefficients $k_i^r(\mu)$. Consider, for example, one of such combinations, occurring in the expression for the mass of charged pions:

$$C_{\pi^\pm} = -\frac{20}{9}[\, k_1^r + k_2^r - k_5^r - \frac{1}{5}(23k_6^r + k_7 + 18k_8^r)\,]$$

Acting by the operator $\mu\frac{d}{d\mu}$ on both sides, we get the $\beta$-functions, or equivalently the $\sigma_i$ coefficients, on the right-hand side. Using Table 2, we see that the quantity $\mu\frac{d}{d\mu}C_{\pi^\pm}$ still remains equal to zero, as it should. Note that we cannot check the gauge invariance of physical quantities, since the expressions for renormalized constants $k_i^r(\mu)$ may contain parts, that do not depend on $\mu$, but may in general depend on the gauge parameter [41].

Finally, we checked that the scale-dependent part of the relations between two- and three-flavor low-energy constants (LECs) [42] , $k_i^r(\mu)$ and $K_i^r(\mu)$, are gauge-independent. Here few comments are in order. The Authors in Ref. [42] provide the matching condition for aforementioned LECs in the Feynman gauge . The so-called two flavor limit of the three-flavor effective theory is considered. More precisely, the external sources in the three-flavor generating functional are restricted to the two-flavor subspace; strange quark mass is much larger than the external momenta and the up and down quark masses $m_s \gg p, m_u, m_d$ . In this limit the three flavor functional reduces to the two flavor functional

$$Z_{two-flavorlimit} = Z_{two-flavor}$$

The both sides reproduce low-energy singularities associated to the propogation of massless pions and photons and the corresponding non-local contributions should cancel. In other words, pions and photons are not relevant for the matching of LECs (only the parts of generating functional, corresponding to heavy particles $\eta$ and $K$ mesons, contribute) and therefore, the relations between $k_i^r(\mu)$ and $K_i^r(\mu)$ remain valid in case of arbitrary gauge parameter. Consider, one of the relations [42]:

$$k_5^r = \frac{6}{5}K_7^r + \frac{1}{5}K_8^r + \frac{4}{9}K_9^r - \frac{1}{5}K_{10}^r - \frac{1}{10}Z\frac{1}{32\pi^2}(\ln\frac{M_K^2}{\mu^2} + 1),$$

where $M_K$ is kaon mass and the coefficients $K_i^r$ was calculated in arbitrary covariant gauge by A.Agadjanov in his thesis. Acting in the same way as in the case of $C_{\pi^\pm}$, we see that this relation does not depend on the gauge parameter.

# 5 Conclusion

We investigated the gauge dependence of the one-loop generating functional for mesons and virtual photons, as well as of $\beta$-functions of the electromagnetic low-energy constants . We faced with the problem of nonapplicability of the conventional heat-kernel method. Therefore, we made calculations,using alternative approach. Finally, we discussed the validity of the obtained result. Namely, we made sure that our result agrees with one obtained by another approach. Then, we showed that the $\beta$-functions do not introduce the dependence of the physical quantities, such as the masses of pions and $\pi\pi$ scattering amplitude,on renormalization scale. Finally, the scale-independence of relations between three- and two-flavour $\beta$-functions was verified.

We plan to analyze how the scale-independent parts of the constants $k_i^r(\mu)$ depend on the gauge parameter. In addition, we want to study the problem in context of lattice QCD calculation of the low-energy constants.

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# Appendices

## A Integrals

In this Appendix we collect the divergent parts of necessary integrals, that are needed during the calculations with help of dimensional regularization. We note that $d = 4 - 2\epsilon$.

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu}{[p^2-2kp+m^2]^3} = \frac{1}{2\Gamma(3)}\frac{i}{16\pi^2\epsilon}g^{\mu\nu} + \text{f. p.},$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu p^\lambda}{[p^2-2kp+m^2]^3} = \frac{1}{2\Gamma(3)}\frac{i}{16\pi^2\epsilon}[g^{\mu\nu}k^\lambda + g^{\mu\lambda}k^\nu + g^{\nu\lambda}k^\mu] + \text{f. p.},$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu p^\lambda p^\rho}{[p^2-2kp+m^2]^3} = \frac{1}{2\Gamma(3)}\frac{i}{16\pi^2\epsilon}\left\{\frac{1}{2}[g^{\mu\nu}k^\rho k^\lambda + g^{\mu\lambda}k^\rho k^\nu + g^{\nu\lambda}k^\mu k^\rho + g^{\mu\rho}k^\nu k^\lambda + \right.$$

$$\left. + g^{\nu\rho}k^\mu k^\lambda + g^{\lambda\rho}k^\mu k^\nu] - \frac{1}{4}(m^2-k^2)[g^{\mu\nu}g^{\rho\lambda} + g^{\mu\lambda}g^{\rho\nu} + g^{\nu\lambda}g^{\mu\rho}]\right\} + \text{f. p.},$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu p^\lambda p^\rho}{[p^2-2kp+m^2]^4} = \frac{1}{4\Gamma(4)}\frac{i}{16\pi^2\epsilon}[g^{\mu\nu}g^{\rho\lambda} + g^{\mu\lambda}g^{\rho\nu} + g^{\nu\lambda}g^{\mu\rho}] + \text{f. p.},$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu p^\lambda p^\rho p^\sigma}{[p^2-2kp+m^2]^4} = \frac{1}{4\Gamma(4)}\frac{i}{16\pi^2\epsilon}[(g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\sigma\nu} + g^{\nu\rho}g^{\sigma\mu})k^\lambda$$

$$+(g^{\mu\nu}g^{\sigma\lambda} + g^{\mu\lambda}g^{\sigma\nu} + g^{\nu\lambda}g^{\mu\sigma})k^\rho + (g^{\mu\lambda}g^{\sigma\rho} + g^{\mu\rho}g^{\lambda\sigma} + g^{\lambda\rho}g^{\mu\sigma})k^\nu$$

$$+(g^{\nu\lambda}g^{\rho\sigma} + g^{\nu\rho}g^{\sigma\lambda} + g^{\lambda\rho}g^{\sigma\nu})k^\mu + (g^{\mu\nu}g^{\rho\lambda} + g^{\mu\lambda}g^{\rho\nu} + g^{\nu\lambda}g^{\mu\rho})k^\sigma] + \text{f. p.},$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu p^\lambda p^\rho p^\sigma p^\epsilon}{[p^2-2kp+m^2]^5} = \frac{1}{8\Gamma(5)}\frac{i}{16\pi^2\epsilon}[(g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\sigma\nu} + g^{\nu\rho}g^{\sigma\mu})g^{\lambda\epsilon}$$

$$+(g^{\mu\nu}g^{\sigma\lambda} + g^{\mu\lambda}g^{\sigma\nu} + g^{\nu\lambda}g^{\mu\sigma})g^{\rho\epsilon} + (g^{\mu\lambda}g^{\sigma\rho} + g^{\mu\rho}g^{\lambda\sigma} + g^{\lambda\rho}g^{\mu\sigma})g^{\nu\epsilon}$$

$$+(g^{\nu\lambda}g^{\rho\sigma} + g^{\nu\rho}g^{\sigma\lambda} + g^{\lambda\rho}g^{\sigma\nu})g^{\mu\epsilon} + (g^{\mu\nu}g^{\rho\lambda} + g^{\mu\lambda}g^{\rho\nu} + g^{\nu\lambda}g^{\mu\rho})g^{\sigma\epsilon}] + \text{f. p.},$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu p^\lambda p^\rho p^\sigma p^\epsilon k^\alpha k^\beta}{[p^2-2kp+m^2]^6} = \frac{1}{16\Gamma(6)}\frac{i}{16\pi^2\epsilon}[$$

$$(g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\sigma\nu} + g^{\nu\rho}g^{\sigma\mu})(g^{\alpha\beta}g^{\lambda\epsilon} + g^{\alpha\lambda}g^{\beta\epsilon} + g^{\alpha\epsilon}g^{\beta\lambda})$$

$$+(g^{\mu\nu}g^{\sigma\lambda} + g^{\mu\lambda}g^{\sigma\nu} + g^{\nu\lambda}g^{\mu\sigma})(g^{\alpha\beta}g^{\rho\epsilon} + g^{\alpha\rho}g^{\beta\epsilon} + g^{\alpha\epsilon}g^{\beta\rho})$$

$$+(g^{\mu\lambda}g^{\sigma\rho} + g^{\mu\rho}g^{\lambda\sigma} + g^{\lambda\rho}g^{\mu\sigma})(g^{\alpha\beta}g^{\nu\epsilon} + g^{\alpha\nu}g^{\beta\epsilon} + g^{\alpha\epsilon}g^{\beta\nu})$$

$$+(g^{\nu\lambda}g^{\rho\sigma} + g^{\nu\rho}g^{\sigma\lambda} + g^{\lambda\rho}g^{\sigma\nu})(g^{\alpha\beta}g^{\mu\epsilon} + g^{\alpha\mu}g^{\beta\epsilon} + g^{\alpha\epsilon}g^{\beta\mu})$$

$$+(g^{\mu\nu}g^{\rho\lambda} + g^{\mu\lambda}g^{\rho\nu} + g^{\nu\lambda}g^{\mu\rho})(g^{\alpha\beta}g^{\sigma\epsilon} + g^{\alpha\sigma}g^{\beta\epsilon} + g^{\alpha\epsilon}g^{\beta\sigma})$$

$$+(g^{\alpha\sigma}g^{\beta\rho}+g^{\beta\sigma}g^{\alpha\rho})(g^{\mu\nu}g^{\epsilon\lambda}+g^{\mu\epsilon}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\epsilon})$$
$$+(g^{\epsilon\sigma}g^{\alpha\rho}+g^{\alpha\sigma}g^{\epsilon\rho})(g^{\mu\nu}g^{\beta\lambda}+g^{\mu\beta}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\beta})$$
$$+(g^{\epsilon\sigma}g^{\beta\rho}+g^{\beta\sigma}g^{\epsilon\rho})(g^{\mu\nu}g^{\alpha\lambda}+g^{\mu\alpha}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\alpha})$$
$$+(g^{\alpha\lambda}g^{\beta\nu}+g^{\beta\lambda}g^{\alpha\nu})(g^{\mu\rho}g^{\epsilon\sigma}+g^{\mu\epsilon}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\epsilon})$$
$$+(g^{\epsilon\lambda}g^{\alpha\nu}+g^{\alpha\lambda}g^{\epsilon\nu})(g^{\mu\rho}g^{\beta\sigma}+g^{\mu\beta}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\beta})$$
$$+(g^{\epsilon\lambda}g^{\beta\nu}+g^{\beta\lambda}g^{\epsilon\nu})(g^{\mu\rho}g^{\alpha\sigma}+g^{\mu\alpha}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\alpha})$$
$$+(g^{\alpha\lambda}g^{\beta\mu}+g^{\beta\lambda}g^{\alpha\mu})(g^{\nu\rho}g^{\epsilon\sigma}+g^{\nu\sigma}g^{\rho\epsilon})+(g^{\epsilon\lambda}g^{\alpha\mu}+g^{\alpha\lambda}g^{\epsilon\mu})(g^{\nu\rho}g^{\beta\sigma}+g^{\nu\sigma}g^{\rho\beta})$$
$$+(g^{\epsilon\lambda}g^{\beta\mu}+g^{\beta\lambda}g^{\epsilon\mu})(g^{\nu\rho}g^{\alpha\sigma}+g^{\nu\sigma}g^{\rho\alpha})+(g^{\alpha\mu}g^{\beta\nu}+g^{\beta\mu}g^{\alpha\nu})(g^{\lambda\rho}g^{\epsilon\sigma}+g^{\lambda\sigma}g^{\rho\epsilon})$$
$$+(g^{\epsilon\mu}g^{\alpha\nu}+g^{\alpha\mu}g^{\epsilon\nu})(g^{\lambda\rho}g^{\beta\sigma}+g^{\lambda\sigma}g^{\rho\beta})+(g^{\epsilon\mu}g^{\beta\nu}+g^{\beta\mu}g^{\epsilon\nu})(g^{\lambda\rho}g^{\alpha\sigma}+g^{\lambda\sigma}g^{\rho\alpha})]+\text{f. p.}$$

# B Calculation of traces

Below, we present the details of calculations.

## B.1 $\mathrm{Tr}(\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta)$

We expand the second trace:

$$\begin{aligned}\mathrm{Sp}\{D_0^{-1}\delta D_0^{-1}\delta\} &= (D_0^{-1})_{ab}(\delta)^{bc}(D_0^{-1})_{cd}(\delta)^{da}+2(D_0^{-1})_{ab}(\delta)^{b\sigma}(D_0^{-1})_{\sigma\rho}(\delta)^{\rho a}\\ &\quad+(D_0^{-1})_{\sigma\rho}(\delta)^{\rho\mu}(D_0^{-1})_{\mu\nu}(\delta)^{\nu\sigma}\end{aligned}\tag{B.1}$$

Analogously, we separate the divergent terms, that depend on the gauge parameter (through $\Delta_{\mu\nu}(x)$), and hereby obtain for $\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta)$:

$$\begin{aligned}\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta) &= \mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta)^{a=1}\\ &\quad+2\int dxdydzdu\,\Delta(x-y)\Delta_{\sigma\rho}(z-u)\langle y|(\delta)^{a\sigma}|z\rangle\langle u|(\delta)^{\rho a}|x\rangle\\ &\quad-2g_{\sigma\rho}\int dxdydzdu\,\Delta(x-y)\Delta_{\mu\nu}(z-u)\langle y|(\delta)^{\rho\mu}|z\rangle\langle u|(\delta)^{\nu\sigma}|x\rangle\\ &\quad+\int dxdydzdu\,\Delta_{\sigma\rho}(x-y)\Delta_{\mu\nu}(z-u)\langle y|(\delta)^{\rho\mu}|z\rangle\langle u|(\delta)^{\nu\sigma}|x\rangle\\ &= \mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta)^{a=1}+2I+2K+L,\end{aligned}\tag{B.2}$$

Thus, it is necessary to find the divergent parts of the integrals $I$, $K$, $L$. The integral $I$ is

$$\begin{aligned}I &= \int dxdydzdu\,\Delta(x-y)\Delta_{\sigma\rho}(z-u)[2Y_\mu(y)\partial_y^\mu\delta(y-z)+c(y)\delta(y-z)]^{a\sigma}\times\\ &\quad\times[2Y_\nu(u)\partial_u^\nu\delta(u-x)+c(u)\delta(u-x)]^{\rho a}\\ &= \int dydu\,[-2Y_\mu(y)\partial_y^\mu\Delta(u-y)+b(y)\Delta(u-y)]^{a\sigma}\times\\ &\quad\times[-2Y_\nu(u)\partial_u^\nu\Delta_{\sigma\rho}(y-u)+b(u)\Delta_{\sigma\rho}(y-u)]^{\rho a}=I_1+I_2+I_3+I_4,\end{aligned}\tag{B.3}$$

where the following matrix is introduced:

$$b(x) = Y_\mu Y^\mu + \Lambda - (\partial_\mu Y^\mu). \tag{B.4}$$

as well as notations for the integrals:

$$\begin{aligned}
I_1 &= 4\int dydu\, Y^{a\sigma}_\mu(y) Y^{\rho a}_\nu(u) I^\mu_{1\,\sigma\rho}, \\
I_2 &= -2\int dydu\, Y^{a\sigma}_\mu(y) b^{\rho a}(u) I^{\mu\nu}_{2\,\sigma\rho}, \\
I_3 &= -2\int dydu\, Y^{a\sigma}_\mu(y) b^{\rho a}(u) I^{\mu}_{3\,\sigma\rho}, \\
I_4 &= \int dydu\, b^{a\sigma}(y) b^{\rho a}(u) I_{4\,\sigma\rho},
\end{aligned} \tag{B.5}$$

with

$$\begin{aligned}
I_1^{\mu\nu\sigma\rho} &= \partial^\mu_y \Delta(u-y) \partial^\nu_u \Delta^{\sigma\rho}(y-u), \\
I_2^{\mu\sigma\rho} &= \partial^\mu_y \Delta(u-y) \Delta^{\sigma\rho}(y-u), \\
I_3^{\mu\sigma\rho} &= \Delta(u-y) \partial^\mu_u \Delta^{\sigma\rho}(y-u), \\
I_4^{\sigma\rho} &= \Delta(u-y) \Delta^{\sigma\rho}(y-u).
\end{aligned} \tag{B.6}$$

The Lorentz indices are raised and lowered by the metric tensor $g^{\mu\nu}$. With the help of Eq.(52) we write the integrals (B.6) in momentum space. For the first integral we have

$$I_1^{\mu\nu\sigma\rho} = (a-1)\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{k_1^\nu k_1^\sigma k_1^\rho k_2^\mu}{k_1^4 k_2^2} e^{-i(k_2-k_1)(u-y)}.$$

Introducing new integration variables

$$(k_1, k_2) \mapsto (k_1, p): \quad p = k_2 - k_1,$$

with the transformation Jacobian $J = 1$ and Feynman parametrization, the integral $I_1^{\mu\nu\sigma\rho}$ takes the form

$$I_1^{\mu\nu\sigma\rho} = (a-1)\int \frac{d^dp}{(2\pi)^d} e^{-ip(u-y)} F_1^{\mu\nu\sigma\rho},$$

where

$$F_1^{\mu\nu\sigma\rho} = 2!\int_0^1 dx \int_0^x dy \int \frac{d^dk_1}{(2\pi)^d} \frac{k_1^\nu k_1^\sigma k_1^\rho p^\mu + k_1^\nu k_1^\sigma k_1^\rho k_1^\mu}{[k_1^2 + 2(1-x)(k_1p) + p^2(1-x)]^3}.$$

Here and below we use the values of integrals, presented in appendix A; we obtain $F_1^{\mu\nu\sigma\rho}$:

$$\begin{aligned}
F_1^{\mu\nu\sigma\rho} = \frac{1}{12}\frac{i}{16\pi^2\epsilon}\Big\{ &\frac{1}{2}[-g^{\nu\sigma}p^\mu p^\rho - g^{\nu\rho}p^\mu p^\sigma - g^{\sigma\rho}p^\mu p^\nu + g^{\mu\nu}p^\sigma p^\rho + g^{\mu\sigma}p^\nu p^\rho + g^{\mu\rho}p^\nu p^\sigma] \\
&-\frac{1}{4}[g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\sigma\nu} + g^{\nu\rho}g^{\sigma\mu}]\Big\} + \text{f. p.}
\end{aligned} \tag{B.7}$$

Then, integrating over the momenta $p$ with help of the formulas

$$\int \frac{d^dp}{(2\pi)^d} p^\mu p^\nu e^{-ip(u-y)} = -\partial_u^\mu \partial_u^\nu \delta(u-y), \quad \int \frac{d^dp}{(2\pi)^d} p^2 e^{-ip(u-y)} = -\partial_u^2 \delta(u-y),$$

we get $I_1^{\mu\nu\sigma\rho}$ and consequently $I_1$, which reads

$$I_1 = -2(a-1)\frac{i}{16\pi^2\epsilon}\int dx X^a \partial^2 X^a + \text{f. p.},$$

where the result for $I_1$ was simplified by summation over Lorentz indices and the use of property of the matrix $Y_\mu$: $Y_\mu^{a\sigma} = -Y_\mu^{\sigma a} = X^a \delta_\mu^\sigma$; the quantity $X^a$ is defined in Eq. (**??**). In the same manner we calculate the divergent parts of the other integrals in Eq. (B.6). We obtain

$$\begin{aligned} I_2 &= -\frac{1}{2}(a-1)\frac{i}{16\pi^2\epsilon}\int dx X^a \partial_\rho b^{\rho a} + \text{f. p.}, \\ I_3 &= -(a-1)\frac{i}{16\pi^2\epsilon}\int dx X^a \partial_\sigma b^{a\sigma} + \text{f. p.}, \\ I_4 &= -\frac{1}{4}(a-1)\frac{i}{16\pi^2\epsilon}\int dx g_{\sigma\rho} b^{a\sigma} b^{\rho a} + \text{f. p.} \end{aligned} \quad \text{(B.8)}$$

The values of integrals from appendix A as well as the following additional formula were used:

$$\int \frac{d^dp}{(2\pi)^d} p^\mu e^{-ip(u-y)} = i\partial_u^\mu \delta(u-y).$$

The final expression for the integral $I = I_1 + I_2 + I_3 + I_4$ can be reduced to a more simple one, if we replace $b^{a\sigma}$ by $b^{\sigma a}$. As it follows from the definition of the matrix $b(x)$,

$$b^{a\sigma} = b^{\sigma a} - 2\partial^\mu Y_\mu^{a\sigma} = b^{\sigma a} - 2\partial^\sigma X^a.$$

Therefore, we get

$$I = -(a-1)\frac{i}{16\pi^2\epsilon}\int dx \left\{-2b^{\rho a}\partial_\rho X^a + \frac{1}{4} g_{\sigma\rho} b^{\sigma a} b^{\rho a}\right\} + \text{f. p.}$$

The second integral $K$ with help of the Eq. (53), takes the form

$$\begin{aligned} K &= -g_{\sigma\rho}\int dxdydzdu\, \Delta(x-y)\Delta_{\mu\nu}(z-u) c^{\rho\mu}(y) c^{\nu\sigma}(u)\delta(y-z)\delta(u-x) \\ &= -g_{\sigma\rho}\int dydu\, \Delta(u-y)\Delta_{\mu\nu}(y-u) c^{\rho\mu}(y) c^{\nu\sigma}(u). \end{aligned} \quad \text{(B.9)}$$

To calculate its divergent part we write

$$K^{\mu\nu} = I_4^{\mu\nu} = -2!(a-1)\int \frac{d^dp}{(2\pi)^d} e^{-ip(u-y)} \int_0^1 dx \int_0^x dy \int \frac{d^dk_1}{(2\pi)^d} \frac{k_1^\mu k_1^\nu}{[k_1^2 + 2(1-x)(k_1 p) + p^2(1-x)]^3},$$

$$K^{\mu\nu} = -\frac{1}{4}(a-1)\frac{i}{16\pi^2\epsilon}g^{\mu\nu}\delta(u-y) + \mathrm{f.\,p.}$$

Then,

$$K = \frac{1}{4}(a-1)\frac{i}{16\pi^2\epsilon}\int dx g_{\mu\nu}g_{\sigma\rho}c^{\rho\mu}(x)c^{\nu\sigma}(x) + \mathrm{f.\,p.}$$

The matrix element $c^{\rho\sigma}$ is

$$c^{\rho\sigma} = -(\rho + X^a X^a)g^{\rho\sigma},$$

and thus we obtain

$$K = (a-1)\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^a X^a)^2 + \mathrm{f.\,p.}$$

The last integral $L$ can be written as

$$L = \int dydu\, c^{\rho\mu}(y)c^{\nu\sigma}(u)L_{\sigma\rho\mu\nu},$$

where

$$L^{\sigma\rho\mu\nu} = \Delta^{\sigma\rho}(u-y)\Delta^{\mu\nu}(y-u)$$

We have

$$L^{\sigma\rho\mu\nu} = (a-1)^2\int\frac{d^dp}{(2\pi)^d}e^{-ip(u-y)}F^{\sigma\rho\mu\nu},$$

with

$$\begin{aligned} F^{\sigma\rho\mu\nu} &= \int\frac{d^dk_1}{(2\pi)^d}\frac{k_1^\mu k_1^\nu(k_1^\sigma + p^\sigma)(k_1^\rho + p^\rho)}{k_1^4(k_1+p)^4} \\ &= 3!\int_0^1 dx\int_0^x dy\int_0^y dz\int\frac{d^dk_1}{(2\pi)^d}\frac{k_1^\mu k_1^\nu k_1^\sigma k_1^\rho}{[k_1^2 + 2(1-y)(k_1p) + p^2(1-y)]^4} + \mathrm{f.\,p.} \end{aligned} \quad \text{(B.10)}$$

Taking the divergent part of the integral in momentum space we get $L^{\sigma\rho\mu\nu}$:

$$L^{\sigma\rho\mu\nu} = \frac{1}{24}(a-1)^2\frac{i}{16\pi^2\epsilon}[g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}]\delta(u-y) + \mathrm{f.\,p.}$$

The integral $L$ becomes

$$L = \frac{1}{24}(a-1)^2\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^a X^a)^2 g_{\mu\nu}g_{\sigma\rho}[g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}] + \mathrm{f.\,p.}$$

$$L = (a-1)^2\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^a X^a)^2 + \mathrm{f.\,p.}$$

Thus, the divergent part of the second trace is

$$\begin{aligned} &div\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta) = div\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta)^{a=1} \\ &-2(a-1)\frac{i}{16\pi^2\epsilon}\int dx\left\{-2b^{\rho a}\partial_\rho X^a + \frac{1}{4}g_{\sigma\rho}b^{\sigma a}b^{\rho a}\right\} \\ &+\left\{2(a-1) + (a-1)^2\right\}\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^a X^a)^2, \end{aligned} \quad \text{(B.11)}$$

where *div* means the divergent part.

## B.2 $\mathrm{Tr}(\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta)$

Next we consider the third trace $\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)$. We have:

$$
\begin{aligned}
&\mathrm{Sp}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) = (D_0^{-1})_{ab}(\delta)^{bc}(D_0^{-1})_{cd}(\delta)^{de}(D_0^{-1})_{ef}(\delta)^{fa} \\
&+3(D_0^{-1})_{ab}(\delta)^{bc}(D_0^{-1})_{cd}(\delta)^{d\sigma}(D_0^{-1})_{\sigma\rho}(\delta)^{\rho a} + (D_0^{-1})_{\sigma\rho}(\delta)^{\rho\lambda}(D_0^{-1})_{\lambda\mu}(\delta)^{\mu a}(D_0^{-1})_{ab}(\delta)^{b\sigma} \\
&+(D_0^{-1})_{\sigma\rho}(\delta)^{\rho\lambda}(D_0^{-1})_{\lambda\mu}(\delta)^{\mu\nu}(D_0^{-1})_{\nu\epsilon}(\delta)^{\epsilon\sigma} \qquad \text{(B.12)}
\end{aligned}
$$

We omit the last term, since it produces finite integral, due to Eq. (53). Then,

$$
\begin{aligned}
&\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) - \mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)^{a=1} = 3M + 3N + \mathrm{f.\,p.} \\
&= 3\int dxdydzdtdudv\, \Delta(x-y)\Delta(z-t)\Delta_{\sigma\rho}(u-v)\langle y|(\delta)^{ab}|z\rangle\langle t|(\delta)^{b\sigma}|u\rangle\langle v|(\delta)^{\rho a}|x\rangle \\
&+3\int dxdydzdtdudv\, \{-\Delta(x-y)\Delta_{\lambda\mu}(z-t)\Delta(u-v)g_{\sigma\rho} - \Delta(x-y)_{\sigma\rho}\Delta(z-t)\Delta(u-v)g_{\lambda\mu} \\
&+\Delta_{\sigma\rho}(x-y)\Delta_{\lambda\mu}(z-t)\Delta(u-v)\}\langle y|(\delta)^{\rho\lambda}|z\rangle\langle t|(\delta)^{\mu a}|u\rangle\langle v|(\delta)^{a\sigma}|x\rangle + \mathrm{f.\,p.} \qquad \text{(B.13)}
\end{aligned}
$$

The integral $M$ is of the form

$$
\begin{aligned}
&M = \int dydtdv\, [-2Y_\mu(y)\partial_y^\mu\Delta(v-y) + b(y)\Delta(v-y)]^{ab} \times \\
&\times[-2Y_\nu(t)\partial_t^\nu\Delta(y-t) + b(t)\Delta(y-t)]^{b\sigma}[-2Y_\lambda(v)\partial_t^\lambda\Delta_{\sigma\rho}(t-v) + b(v)\Delta_{\sigma\rho}(t-v)]^{\rho a} \\
&= M_1 + M_2 + M_3 + M_4 + \mathrm{f.\,p.}, \qquad \text{(B.14)}
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &= -8\int dydtdv\, Y_\mu^{ab}(y)Y_\nu^{b\sigma}(t)Y_\lambda^{\rho a}(v)M_{1\sigma\rho}^{\mu\nu\lambda}, \\
M_2 &= 4\int dydtdv\, Y_\mu^{ab}(y)Y_\nu^{b\sigma}(t)b^{\rho a}(v)M_{2\sigma\rho}^{\mu\nu}, \\
M_3 &= 4\int dydtdv\, Y_\mu^{ab}(y)Y_\nu^{\rho a}(v)b^{b\sigma}(t)M_{3\sigma\rho}^{\mu\nu}, \\
M_4 &= 4\int dydtdv\, Y_\mu^{b\sigma}(t)Y_\nu^{\rho a}(v)b^{ab}(y)M_{4\sigma\rho}^{\mu\nu}, \qquad \text{(B.15)}
\end{aligned}
$$

with

$$
\begin{aligned}
M_1^{\mu\nu\lambda\sigma\rho} &= \partial_y^\mu\Delta(v-y)\partial_t^\nu\Delta(y-t)\partial_v^\lambda\Delta^{\sigma\rho}(t-v), \\
M_2^{\mu\nu\sigma\rho} &= \partial_y^\mu\Delta(v-y)\partial_t^\nu\Delta(y-t)\Delta^{\sigma\rho}(t-v), \\
M_3^{\mu\nu\sigma\rho} &= \partial_y^\mu\Delta(v-y)\Delta(y-t)\partial_v^\nu\Delta^{\sigma\rho}(t-v), \\
M_4^{\mu\nu\sigma\rho} &= \Delta(v-y)\partial_t^\mu\Delta(y-t)\partial_v^\nu\Delta^{\sigma\rho}(t-v). \qquad \text{(B.16)}
\end{aligned}
$$

To calculate the divergent part of $M_1^{\mu\nu\lambda\sigma\rho}$, we write

$$M_1^{\mu\nu\lambda\sigma\rho} = i^3(a-1)\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\frac{k_1^\mu k_2^\nu k_3^\lambda k_3^\sigma k_3^\rho}{k_1^2k_2^2k_3^4}e^{-ik_1(v-y)-ik_2(y-t)-ik_3(t-v)} \tag{B.17}$$

Then we similarly introduce the new integration variables:

$$(k_1,k_2,k_3)\mapsto(p,q,k): \quad p=k_1-k_3,\ q=k_2-k_3,\ k=k_3,$$

with Jacobian $J=1$, and obtain

$$M_1^{\mu\nu\lambda\sigma\rho} = i^3(a-1)\int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}e^{ip(y-v)+iq(t-y)}G_1^{\mu\nu\lambda\sigma\rho},$$

where

$$G_1^{\mu\nu\lambda\sigma\rho} = 3!\int_0^1 dx\int_0^x dy\int_0^y dz\int\frac{d^dk}{(2\pi)^d}\frac{p^\mu k^\nu k^\lambda k^\sigma k^\rho + q^\nu k^\mu k^\lambda k^\sigma k^\rho + k^\mu k^\nu k^\lambda k^\sigma k^\rho + \text{f. p.}}{[k^2+2k[p(x-y)+q(1-x)]+p^2(x-y+q^2(1-z))]^4}$$

After integrations, we obtain

$$\begin{aligned}G_1^{\mu\nu\lambda\sigma\rho} &= \frac{1}{4}\frac{i}{16\pi^2\epsilon}\frac{1}{24}\{(3p^\mu-q^\mu)(g^{\nu\lambda}g^{\rho\sigma}+g^{\nu\rho}g^{\sigma\lambda}+g^{\lambda\rho}g^{\sigma\nu})\\ &+(3q^\nu-p^\nu)(g^{\mu\lambda}g^{\sigma\rho}+g^{\mu\rho}g^{\lambda\sigma}+g^{\lambda\rho}g^{\mu\sigma})-(p^\lambda+q^\lambda)(g^{\mu\nu}g^{\sigma\rho}+g^{\mu\rho}g^{\sigma\nu}+g^{\nu\rho}g^{\sigma\mu})\\ &-(p^\rho+q^\rho)(g^{\mu\nu}g^{\sigma\lambda}+g^{\mu\lambda}g^{\sigma\nu}+g^{\nu\lambda}g^{\mu\sigma})-(p^\sigma+q^\sigma)(g^{\mu\nu}g^{\rho\lambda}+g^{\mu\lambda}g^{\rho\nu}+g^{\nu\lambda}g^{\mu\rho})\}+\text{f. p.}\end{aligned}$$

Then, integrating over $p,q$ according to the formulas

$$\int\frac{d^dp}{(2\pi)^d}p^\mu e^{ip(y-v)} = -i\partial_y^\mu\delta(y-v), \quad \int\frac{d^dq}{(2\pi)^d}q^\nu e^{iq(t-y)} = i\partial_y^\nu\delta(t-y),$$

we get $M_1^{\mu\nu\lambda\sigma\rho}$. After summation over Lorentz indices and performing of the necessary integrations, the divergent part of $M_1$ reads

$$M_1 = 2(a-1)\frac{i}{16\pi^2\epsilon}\int dx Y_\mu^{ab}X^a\partial^\mu X^b + \text{f. p.}$$

The integral $M_2^{\mu\nu\sigma\rho}$is

$$M_2^{\mu\nu\sigma\rho} = i^2(a-1)\int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}e^{ip(y-v)+iq(t-y)}G_2^{\mu\nu\sigma\rho},$$

where

$$G_2^{\mu\nu\sigma\rho} = 3!\int_0^1 dx\int_0^x dy\int_0^y dz\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu k^\sigma k^\rho}{[k^2+\ldots]^4}+\text{f. p.}$$

Thus,

$$M_2^{\mu\nu\sigma\rho} = -\frac{1}{24}(a-1)\frac{i}{16\pi^2\epsilon}[g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}]\delta(y-v)\delta(t-y) + \text{f. p.}$$

We note that the other integrals $M_3^{\mu\nu\sigma\rho}, M_4^{\mu\nu\sigma\rho}$ have the same divergent part. Therefore, we get

$$\begin{aligned} M_2 + M_3 + M_4 &= -\frac{1}{6}(a-1)\frac{i}{16\pi^2\epsilon}\int dx\left[Y_\mu^{ab}Y_\nu^{b\sigma}b^{\rho a} + Y_\mu^{ab}Y_\nu^{\rho a}b^{b\sigma} + Y_\mu^{b\sigma}Y_\nu^{\rho a}b^{ab}\right]\times \\ &\quad \times[g^{\mu\nu}g_{\sigma\rho} + \delta_\rho^\mu\delta_\sigma^\nu + \delta_\sigma^\mu\delta_\rho^\nu] + \text{f. p.}, \end{aligned} \tag{B.18}$$

$$M_2 + M_3 + M_4 = -\frac{1}{6}(a-1)\frac{i}{16\pi^2\epsilon}\int dx\left\{6Y_\rho^{ab}X^b(b^{\rho a} + b^{a\rho}) - 24b^{ab}X^aX^b\right\} + \text{f. p.}$$

The integral $N$, taking into account Eq. (53), can be written as

$$N = N_1 + N_2 + N_3 + \text{f. p.},$$

where

$$\begin{aligned} N_1 &= -4\int dydtdv\, g_{\sigma\rho}c^{\rho\lambda}(y)Y_\nu^{\mu a}(t)Y_\epsilon^{a\sigma}(v)N_{1\lambda\mu}^{\nu\epsilon}, \\ N_2 &= -4\int dydtdv\, g_{\lambda\mu}c^{\rho\lambda}(y)Y_\nu^{\mu a}(t)Y_\epsilon^{a\sigma}(v)N_{2\sigma\rho}^{\nu\epsilon}, \\ N_3 &= 4\int dydtdv\, c^{\rho\lambda}(y)Y_\nu^{\mu a}(t)Y_\epsilon^{a\sigma}(v)N_{3\sigma\rho\lambda\mu}^{\nu\epsilon}, \end{aligned} \tag{B.19}$$

with

$$\begin{aligned} N_1^{\nu\lambda\mu\epsilon} &= \Delta(v-y)\partial_t^\nu\Delta^{\lambda\mu}(y-t)\partial_v^\epsilon\Delta(t-v), \\ N_2^{\sigma\rho\nu\epsilon} &= \Delta^{\sigma\rho}(v-y)\partial_t^\nu\Delta(y-t)\partial_v^\epsilon\Delta(t-v), \\ N_3^{\sigma\rho\nu\lambda\mu\epsilon} &= \Delta^{\sigma\rho}(v-y)\partial_t^\nu\Delta^{\lambda\mu}(y-t)\partial_v^\epsilon\Delta(t-v). \end{aligned} \tag{B.20}$$

The integrals $N_1^{\nu\epsilon\lambda\mu}$, $N_2^{\sigma\rho\nu\epsilon}$ have the same divergent part as $M_2^{\mu\nu\sigma\rho}$. Thus,

$$\begin{aligned} N_1^{\nu\epsilon\lambda\mu} &= -\frac{1}{24}(a-1)\frac{i}{16\pi^2\epsilon}[g^{\mu\nu}g^{\lambda\epsilon} + g^{\nu\lambda}g^{\mu\epsilon} + g^{\mu\lambda}g^{\nu\epsilon}]\delta(y-v)\delta(t-y) + \text{f. p.}, \\ N_2^{\sigma\rho\nu\epsilon} &= -\frac{1}{24}(a-1)\frac{i}{16\pi^2\epsilon}[g^{\sigma\rho}g^{\nu\epsilon} + g^{\nu\sigma}g^{\rho\epsilon} + g^{\sigma\epsilon}g^{\rho\nu}]\delta(y-v)\delta(t-y) + \text{f. p.} \end{aligned} \tag{B.21}$$

Substituting $c^{\rho\lambda}Y_\nu^{\mu a}Y_\epsilon^{a\sigma} = (\rho + X^aX^a)X^bX^bg^{\rho\lambda}\delta_\nu^\mu\delta_\epsilon^\sigma$, we obtain

$$N_1 + N_2 = 8(a-1)\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^aX^a)X^bX^b + \text{f. p.}$$

The integral $N_3^{\sigma\rho\nu\lambda\mu\epsilon}$ is of the form

$$N_3^{\sigma\rho\nu\lambda\mu\epsilon} = (a-1)^2)\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}e^{ip(y-v)+iq(t-y)}G_3^{\sigma\rho\nu\lambda\mu\epsilon},$$

where

$$G_3^{\sigma\rho\nu\lambda\mu\epsilon} = 4!\int_0^1 dx\int_0^x dy\int_0^y dz\int_0^z du\int\frac{d^dk}{(2\pi)^d}\frac{k^\sigma k^\rho k^\lambda k^\mu k^\nu k^\epsilon}{[k^2+\ldots]^5} + \text{f. p.}$$

Using the value of the integral in appendix A and then noting that

$$g_{\rho\lambda}g_{\mu\nu}g_{\sigma\epsilon}[g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon} + \text{perm.}] = 192$$

we get $N_3$, which reads

$$N_3 = 4(a-1)^2\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^aX^a)X^bX^b + \text{f. p.}$$

Thus, the divergent part of $\text{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)$ is

$$\begin{aligned}
&div\text{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) = div\text{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)^{a=1}\\
&+6(a-1)\frac{i}{16\pi^2\epsilon}\int dxY_\mu^{ab}X^a\partial^\mu X^b\\
&-\frac{1}{2}(a-1)\frac{i}{16\pi^2\epsilon}\int dx\left\{6Y_\rho^{ab}X^b(b^{\rho a}+b^{a\rho}) - 24b^{ab}X^aX^b\right\}\\
&+\left\{24(a-1)+12(a-1)^2\right\}\frac{i}{16\pi^2\epsilon}\int dx(\rho+X^aX^a)X^bX^b. \qquad \text{(B.22)}
\end{aligned}$$

## B.3 $\text{Tr}(\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta)$

Finally, we calculate the divergent part of $\text{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)$. We have:

$$\begin{aligned}
&\text{Sp}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) = (D_0^{-1})_{ab}(\delta)^{bc}(D_0^{-1})_{cd}(\delta)^{de}(D_0^{-1})_{ef}(\delta)^{fg}(D_0^{-1})_{gk}(\delta)^{ka}\\
&+4(D_0^{-1})_{ab}(\delta)^{bc}(D_0^{-1})_{cd}(\delta)^{de}(D_0^{-1})_{ef}(\delta)^{f\sigma}(D_0^{-1})_{\sigma\rho}(\delta)^{\rho a}\\
&+2(D_0^{-1})_{ab}(\delta)^{b\sigma}(D_0^{-1})_{\sigma\rho}(\delta)^{\rho e}(D_0^{-1})_{ef}(\delta)^{f\lambda}(D_0^{-1})_{\lambda\mu}(\delta)^{\mu a} + \text{f. p.}, \qquad \text{(B.23)}
\end{aligned}$$

where we omitted terms, that produce finite integrals. Then

$$\begin{aligned}
&\text{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) - \text{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)^{a=1} = 4P + 2Q + \text{f. p.}\\
&= 4\int dxdydzdtdudvdrds\left\{\Delta(x-y)\Delta(z-t)\Delta(u-v)\Delta_{\sigma\rho}(r-s)\times\right.\\
&\left.\times\langle y|(\delta)^{ab}|z\rangle\langle t|(\delta)^{bc}|u\rangle\langle v|(\delta)^{c\sigma}|r\rangle\langle s|(\delta)^{\rho a}|x\rangle\right\}\\
&+2\int dxdydzdtdudvdrds\left\{-\Delta(x-y)\Delta(z-t)\Delta(u-v)\Delta_{\lambda\mu}(r-s)g_{\sigma\rho}\right.\\
&\left.-\Delta(x-y)\Delta_{\sigma\rho}(z-t)\Delta(u-v)\Delta(r-s)g_{\lambda\mu} + \Delta(x-y)\Delta_{\sigma\rho}(z-t)\Delta(u-v)\Delta_{\lambda\mu}(r-s)\right\}\times\\
&\times\langle y|(\delta)^{a\sigma}|z\rangle\langle t|(\delta)^{\rho b}|u\rangle\langle v|(\delta)^{b\lambda}|r\rangle\langle s|(\delta)^{\mu a}|x\rangle + \text{f. p.} \qquad \text{(B.24)}
\end{aligned}$$

The integral $P$ is of the form

$$P = 16 \int dydtdvds\, Y_\mu^{ab}(y) Y_\nu^{bc}(t) Y_\lambda^{c\sigma}(v) Y_\epsilon^{\rho a}(s) P_{\sigma\rho}^{\mu\nu\lambda\epsilon} + \text{f. p.},$$

where

$$P^{\mu\nu\lambda\epsilon\sigma\rho} = \partial_y^\mu \Delta(s-y) \partial_t^\nu \Delta(y-t) \partial_v^\lambda \Delta(t-v) \partial_s^\epsilon \Delta^{\sigma\rho}(v-s).$$

The integral $P^{\mu\nu\lambda\epsilon\sigma\rho}$ can be written as

$$P^{\mu\nu\lambda\epsilon\sigma\rho} = -(a-1) \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{d^d k_4}{(2\pi)^d} \frac{k_1^\mu k_2^\nu k_3^\lambda k_4^\epsilon k_4^\sigma k_4^\rho}{k_1^2 k_2^2 k_3^2 k_4^4} e^{-ik_1(s-y)-ik_2(y-t)-ik_3(t-v)-ik_4(v-s)}.$$

Introducing the new integration variables

$$(k_1, k_2, k_3, k_4) \mapsto (p, k, q, r): \quad p = k_1 - k_2,\ q = k_3 - k_2,\ r = k_4 - k_3,\ k = k_2,$$

with Jacobian $J = 1$, we get

$$P^{\mu\nu\lambda\epsilon\sigma\rho} = -(a-1) \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} e^{ip(y-s)+iq(s-t)+ir(s-v)} H^{\mu\nu\lambda\epsilon\sigma\rho}$$

where

$$H^{\mu\nu\lambda\epsilon\sigma\rho} = 4! \int_0^1 dx \int_0^x dy \int_0^y dz \int_0^z du \int \frac{d^d k}{(2\pi)^d} \frac{k^\sigma k^\rho k^\lambda k^\mu k^\nu k^\epsilon}{[k^2 + \ldots]^5} + \text{f. p.}$$

The latter integral has the same divergent part as $G_3^{\sigma\rho\nu\lambda\mu\epsilon}$. Thus,

$$H^{\mu\nu\lambda\epsilon\sigma\rho} = \frac{1}{8\Gamma(5)} \frac{i}{16\pi^2\epsilon} \frac{1}{24} [g^{\mu\nu} g^{\sigma\rho} g^{\lambda\epsilon} + \text{perm.}] \delta(y-s)\delta(s-t)\delta(s-v) + \text{f. p.}$$

Substituting $Y_\lambda^{c\sigma} Y_\epsilon^{\rho a} = -X^a X^c \delta_\lambda^\sigma \delta_\epsilon^\rho$ and noting that

$$g_{\sigma\lambda} g_{\rho\epsilon} [g^{\mu\nu} g^{\sigma\rho} g^{\lambda\epsilon} + \text{perm.}] = 48 g^{\mu\nu},$$

we obtain

$$P = 4(a-1) \frac{i}{16\pi^2\epsilon} \int dx g^{\mu\nu} Y_\mu^{ab} Y_\nu^{bc} X^a X^c + \text{f. p.}$$

We write the second integral $Q$ as

$$Q = Q_1 + Q_2 + Q_3 + \text{f. p.},$$

where

$$Q_1 \;=\; -16 \int dydtdvds\, g_{\alpha\beta} Y_\sigma^{a\alpha}(y) Y_\rho^{\beta b}(t) Y_\nu^{b\lambda}(v) Y_\epsilon^{\mu a}(s) Q_{1\lambda\mu}^{\sigma\rho\nu\epsilon} + \text{f. p.},$$

$$
\begin{aligned}
Q_2 &= -16\int dydtdvds\, g_{\alpha\beta} Y^{a\lambda}_{\sigma}(y) Y^{\mu b}_{\rho}(t) Y^{b\alpha}_{\nu}(v) Y^{\beta a}_{\epsilon}(s) Q^{\sigma\rho\nu\epsilon}_{2\lambda\mu} + \text{f. p.}, \\
Q_3 &= 16\int dydtdvds\, Y^{a\sigma}_{\alpha}(y) Y^{\rho b}_{\beta}(t) Y^{b\lambda}_{\nu}(v) Y^{\mu a}_{\epsilon}(s) Q^{\alpha\beta\nu\epsilon}_{3\sigma\rho\lambda\mu} + \text{f. p.},
\end{aligned} \tag{B.25}
$$

with

$$
\begin{aligned}
Q^{\sigma\rho\nu\epsilon\lambda\mu}_1 &= \partial^{\sigma}_y \Delta(s-y)\partial^{\rho}_t \Delta(y-t)\partial^{\nu}_v \Delta(t-v)\partial^{\epsilon}_s \Delta^{\lambda\mu}(v-s), \\
Q^{\sigma\rho\lambda\mu\nu\epsilon}_2 &= \partial^{\sigma}_y \Delta(s-y)\partial^{\rho}_t \Delta^{\lambda\mu}(y-t)\partial^{\nu}_v \Delta(t-v)\partial^{\epsilon}_s \Delta(v-s), \\
Q^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu}_3 &= \partial^{\alpha}_y \Delta(s-y)\partial^{\beta}_t \Delta^{\sigma\rho}(y-t)\partial^{\nu}_v \Delta(t-v)\partial^{\epsilon}_s \Delta^{\lambda\mu}(v-s).
\end{aligned} \tag{B.26}
$$

The integrals $Q^{\sigma\rho\nu\epsilon\lambda\mu}_1, Q^{\sigma\rho\lambda\mu\nu\epsilon}_2$ have the same divergent part as $H^{\mu\nu\lambda\epsilon\sigma\rho}$. Since

$$
g_{\sigma\rho}g_{\lambda\nu}g_{\mu\epsilon}[g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon} + \text{perm.}] = 192, \quad g_{\nu\epsilon}g_{\lambda\sigma}g_{\mu\rho}[g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon} + \text{perm.}] = 192,
$$

we obtain

$$
Q_1 + Q_2 = 32(a-1)\frac{i}{16\pi^2\epsilon}\int dx X^a X^a X^b X^b + \text{f. p.}
$$

The third integral $Q^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu}_3$ is

$$
Q^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu}_3 = (a-1)^2\int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{d^dr}{(2\pi)^d} e^{ip(y-s)+iq(s-t)+ir(s-v)} H^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu}_3,
$$

where

$$
H^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu}_3 = 5!\int_0^1 dx\int_0^x dy\int_0^y dz\int_0^z du\int_0^u dv\int \frac{d^dk}{(2\pi)^d}\frac{k^\alpha k^\beta k^\sigma k^\rho k^\nu k^\epsilon k^\lambda k^\mu}{[k^2+\ldots]^6} + \text{f. p.}
$$

Performing necessary integrations, we get

$$
Q^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu}_3 = (a-1)^2\frac{1}{16}\frac{i}{16\pi^2\epsilon}\frac{1}{120}[g^{\alpha\beta}g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon} + \text{perm.}]\delta(y-s)\delta(s-t)\delta(s-v) + \text{f. p.}
$$

Then we substitute $Y^{a\sigma}_{\alpha}Y^{\rho b}_{\beta}Y^{b\lambda}_{\nu}Y^{\mu a}_{\epsilon} = X^a X^a X^b X^b \delta^{\sigma}_{\alpha}\delta^{\rho}_{\beta}\delta^{\lambda}_{\nu}\delta^{\mu}_{\epsilon}$. Since

$$
g_{\sigma\alpha}g_{\rho\beta}g_{\lambda\nu}g_{\mu\epsilon}[g^{\alpha\beta}g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon} + \text{perm.}] = 1920,
$$

we obtain

$$
Q_3 = 16(a-1)^2\frac{i}{16\pi^2\epsilon}\int dx X^a X^a X^b X^b + \text{f. p.}
$$

Hereby, the divergent part of $\text{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)$ reads

$$
div\text{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) = div\text{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)^{a=1}
$$

$$+16(a-1)\frac{i}{16\pi^2\epsilon}\int dx g^{\mu\nu}Y_\mu^{ab}Y_\nu^{bc}X^aX^c$$
$$+\left\{64(a-1)+32(a-1)^2\right\}\frac{i}{16\pi^2\epsilon}\int dx X^aX^aX^bX^b \tag{B.27}$$

## B.4 Summation over flavour indices

Thus, the divergent part of the one-loop functional in the arbitrary gauge is given by the expression

$$div Z_{\text{one loop}} = div Z^{a=1}_{\text{one loop}} + \frac{1}{2}(1-a)\frac{1}{16\pi^2\epsilon}\int dx \Big\{-2b^{\rho a}\partial_\rho X^a + \frac{1}{4}g_{\sigma\rho}b^{\sigma a}b^{\rho a}$$
$$+2Y_\mu^{ab}X^a\partial^\mu X^b - Y_\rho^{ab}X^b(b^{\rho a}+b^{a\rho}) + 4b^{ab}X^aX^b - 4g^{\mu\nu}Y_\mu^{ab}Y_\nu^{bc}X^aX^c\Big\} \tag{B.28}$$
$$+\left\{2(a-1)+(a-1)^2\right\}\frac{1}{16\pi^2\epsilon}\int dx\left\{\frac{1}{4}(\rho+X^aX^a)^2 - 2(\rho+X^aX^a)X^bX^b + 4X^aX^aX^bX^b\right\}$$

For simplification of the result, we have to sum over flavor indices. This is done with help of the formula, which follows from the completeness relation for the generators $\lambda^a$ of SU($N$):

$$\sum_a \langle A\lambda^a\rangle\langle B\lambda^a\rangle = 2\langle AB\rangle - \frac{2}{N}\langle A\rangle\langle B\rangle. \tag{B.29}$$

We farther set $Q_R = Q_L = Q$. Since $H_L = \langle Q_R - Q_L\rangle = \langle Q - Q\rangle = 0$,

$$X^aX^a = \frac{F_0^2}{16}\langle H_L\lambda^a\rangle\langle H_L\lambda^a\rangle = \frac{F_0^2}{8}\langle H_L^2\rangle.$$

Then, $\rho + X^aX^a = \frac{1}{2}F_0^2\langle H_L^2\rangle$ and the second integral in Eq. (B.28) is equal to zero:

$$\frac{1}{4}(\rho+X^aX^a)^2 - 2(\rho+X^aX^a)X^bX^b + 4X^aX^aX^bX^b = \left(\frac{1}{16}-\frac{1}{8}+\frac{1}{16}\right)F_0^2\langle H_L^2\rangle = 0.$$

From the definition of the matrix $b$ it follows that

$$\begin{aligned} b^{\rho a} &= X_\mu^{\rho c}\Gamma^{\mu ca} + \Lambda^{\rho a} - \partial^\mu X_\mu^{\rho a}, \quad b^{\rho a} + b^{a\rho} = 2(X_\mu^{\rho c}\Gamma^{\mu ca} + \Lambda^{\rho a}), \\ b^{ab} &= \Gamma_\mu^{ac}\Gamma^{\mu cb} + 4X^aX^b + \Lambda^{ab} - \partial^\mu\Gamma_\mu^{ab}. \end{aligned} \tag{B.30}$$

Then,

$$4b^{ab}X^aX^b - 4g^{\mu\nu}Y_\mu^{ab}Y_\nu^{bc}X^aX^c = 4X^aX^b\Big\{\frac{1}{2}\langle[\Delta_\mu,\lambda^a][\Delta^\mu,\lambda^b]\rangle + \frac{1}{4}\langle\{\lambda^a,\lambda^b\}\sigma\rangle$$
$$-\frac{C}{8F_0^2}(\langle[H_R+H_L,\lambda^a][H_R-H_L,\lambda^b] + a\leftrightarrow b\rangle)\Big\} \tag{B.31}$$

The matrix element $b^{\rho a}$ can be written as

$$b^{\rho a} = -\frac{1}{2} F_0 \langle ([H_R, \Delta^\rho] + D^\rho H_L)\, \lambda^a \rangle,$$

since $X_\mu^{\rho b} \Gamma^{\mu ba} = -\frac{1}{4} F_0 \langle \lambda^a [\Gamma^\rho, H_L] \rangle$. Using Eq. (B.29), we obtain:

$$\begin{aligned}
& b^{\rho a} \partial_\rho X^a = \frac{1}{4} F_0^2 \langle \partial_\mu H_L ([H_R, \Delta^\mu] + D^\mu H_L) \rangle, \\
& g_{\sigma\rho} b^{\sigma a} b^{\rho a} = \frac{1}{2} F_0^2 \langle ([H_R, \Delta_\mu] + D_\mu H_L)([H_R, \Delta^\mu] + D^\mu H_L) \rangle, \\
& \Gamma_\mu^{ab} X^a \partial^\mu X^b = -\frac{1}{8} F_0^2 \langle \partial^\mu H_L [\Gamma_\mu, H_L] \rangle, \\
& \Gamma_\rho^{ab} X^b X^c \Gamma^{\rho ca} = -\frac{1}{8} F_0^2 \langle [\Gamma^\mu, H_L][\Gamma_\mu, H_L] \rangle, \\
& \Gamma_\rho^{ab} X^b \Lambda^{\rho a} = \frac{1}{4} F_0^2 \langle [\Gamma_\mu, H_L] \left( [H_R, \Delta^\mu] + \frac{1}{2} D^\mu H_L \right) \rangle, \\
& X^a X^b \langle [\Delta_\mu, \lambda^a][\Delta^\mu, \lambda^b] \rangle = \frac{1}{4} F_0^2 \langle [H_L, \Delta_\mu][H_L, \Delta^\mu] \rangle, \\
& X^a X^b \langle \{\lambda^a, \lambda^b\} \sigma \rangle = \frac{1}{2} F_0^2 \langle H_L^2 \sigma \rangle, \\
& X^a X^b (\langle [H_R + H_L, \lambda^a][H_R - H_L, \lambda^b] + a \leftrightarrow b \rangle) = \frac{1}{8} F_0^2 \langle [H_R + H_L, H_R - H_L]^2 \rangle
\end{aligned} \tag{B.32}$$

We also note that $D_\mu H_L$ can be written as

$$D_\mu H_L = [H_R, \Delta_\mu] + G_\mu,$$

where

$$G^\mu = u^+ c_R^\mu Q u - u c_L^\mu Q u.$$

After substitution of these formulas into Eq. (B.28), the divergent part of the one-loop functional takes the form

$$\begin{aligned}
& div Z_{\text{one loop}} = div Z_{\text{one loop}}^{a=1} + \frac{1}{4} F_0^2 (1-a) \frac{1}{16\pi^2 \epsilon} \int dx \Big\{ \langle [H_L, \Delta_\mu]^2 \rangle - \langle [H_R, \Delta_\mu]^2 \rangle \\
& + \langle H_L^2 \sigma \rangle - 2 \langle [H_R, \Delta_\mu] G^\mu \rangle - \frac{3}{4} \langle G_\mu G^\mu \rangle - \frac{1}{8} F_0^2 Z \langle [H_R + H_L, H_R - H_L]^2 \rangle \Big\},
\end{aligned} \tag{B.33}$$

where $Z = C/F_0^4$.