St. Andrew the First-called Georgian University of the Patriarchate of Georgia
Faculty of Natural Sciences

Master of Science Thesis

One-loop generating functional for mesons and virtual photons

by

Andria Agadjanov

Project leader: Prof. Anzor Khelashvili
Supervisor: Priv. Doz. Dr. Akaki Rusetsky

Tbilisi, 2011

Abstract

At low energies, the effective field theory of Quantum Chromodynamics and Quantum Electrodynamics (QCD+QED), called Chiral Perturbation Theory for mesons and virtual photons, allows to evaluate physical quantities in perturbative manner. In present thesis, the divergent part of the corresponding one-loop generating functional is calculated in case of an arbitrary covariant gauge. These calculations provide a deeper insight on the low-energy effective theory of QCD+QED. The β-functions in the three-flavour case are obtained. The independence of various physical quantities on the renormalization scale is verified. Comparison of the β-functions with the ones given in the literature is made. The obtained results might be used for the extraction of the low-energy constants from the lattice QCD data.

Contents

1 Introduction

Quantum Chromodynamics (QCD) is regarded as a theory of strong interactions. It describes interactions between quarks, which are mediated by gluons. QCD is asymptotically free theory, which means that at quite high energies, the strength of interactions between quarks, i.e. coupling constant, goes to zero. This allows one to use QCD in perturbative manner and obtain sensible theoretical predictions. However, quarks are not observed as free particles in Nature. This is due to the phenomenon, known as confinement. At low energies, quarks rather form bound states, called hadrons.

One cannot describe interactions between hadrons with the help of QCD, since its coupling constant becomes large. Therefore, the question arises, whether it is possible to construct some effective field theory, which would replace QCD at low energies. The answer is follows.

In the chiral limit where the light up, down and strange quark masses go to zero, the QCD Lagrangian has a $\mathrm{SU}(3)_R \times \mathrm{SU}(3)_L$ chiral symmetry that is spontaneously broken to $\mathrm{SU}(3)_{R+L}$ symmetry. The eight Goldstone bosons arise, that can be identified with the eight lightest hadrons (mesons): the π's, K's and η. Their interactions are described by an effective low energy theory, called Chiral Perturbation Theory (ChPT).

ChPT contains all terms allowed by the symmetry of the QCD Lagrangian in the chiral limit. At low momenta the chiral Lagrangian can be expanded in derivatives of the Goldstone fields and in the masses of the three light quarks. ChPT is a nonrenormalizable theory in usual sense. However, as long as one includes every possible interaction terms allowed by symmetries, the nonrenormalizability is not a problem: the theory is renormalizable order by order. ChPT can be used perturbatively, since every loop and the associated counterterm correspond to successively higher powers of momenta or quark masses. Therefore, at low energies, the contributions from higher loops are small.

Since some of the mesons are electrically charged, it is important to include electromagnetic interactions in ChPT framework. The obtained theory (ChPT with virtual photons) is an effective theory of QCD+QED.

Next we consider ChPT for mesons and virtual photons at one-loop (next-to-leading order). Loops, in which mesons and photons run, produce ultraviolet divergences, which can be absorbed by introducing additional counterterms. From the point of view of path integral formulation, these divergences are contained in the so-called one-loop generating functional. The counterterms in the Lagrangian contain the so-called low-energy constants (LECs), which are also divergent. Choosing their divergent parts appropriately, we get finite generating functional of ChPT at next-to-leading order.

Thus, one has to calculate the divergent part of the one-loop functional. This has already been done in the literature. As it is known, after quantization of electromagnetic field, the gauge fixing term appears in Lagrangian. It depends on a gauge parameter a. However, the calculations were done in the Feynman gauge, where $a = 1$. In present thesis we extend the evaluation of the divergent part of the one-loop functional to the case of arbitrary gauge. To that end we use other method of calculation, since for $a \neq 1$, the differential operator in the Lagrangian turns out to be of a so-called "non-minimal" type and the conventional heat-kernel method is no more applicable. Further, we obtain the dependence of the β-functions of the different low-energy constants on the gauge parameter. Then we check the validity of the result and compare our β-functions with the (incomplete) results, available in the literature.

We think, that the present calculations provide a deeper insight on the low-energy effective theory of QCD+QED. In addition, our results can be used for the extraction of the so-called electromagnetic low-energy constants from the lattice QCD data, since the Feynman gauge is not the most convenient one to perform lattice Monte-Carlo simulations.

2 Basics of Chiral Perturbation Theory

2.1 QCD and Chiral Symmetry

Quantum Chromodynamics (QCD) is the gauge theory of the strong interactions [1, 2, 3]. Its gauge group is color SU(3) group. The matter fields of QCD are spin-half fermion particles, called quarks with six different flavors in addition to their three possible colors.

Thus, the QCD Lagrangian can be obtained from the gauge principle and it reads

$$\mathcal{L}_{qcd} = \sum_f \bar{q_f}(i\not{D} - m_f)q_f - \frac{1}{4}G_{\mu\nu,a}G^{\mu\nu,a}, \tag{1}$$

where summation is done over all six quark flavors. The quark field q_f consists of a color triplet and $G_{\mu\nu,a}$ denotes gauge field strength tensor:

$$G_{\mu\nu,a} = \partial_\mu G_{\nu,a} - \partial_\nu G_{\mu,a} + g f_{abc} G_{\mu,b} G_{\nu,c},$$

where $G_{\mu,a}$ is gauge potential and g coupling constant between quarks and gauge fields (or, in other words, gluons). The covariant derivative D_μ has the form

$$D_\mu = \partial_\mu - ig \sum_{a=1}^{8} \frac{\lambda_a^C}{2} G_{\mu,a};$$

here λ_a^C are Gell-Mann matrices, which act in color space (C superscript shows it). The existence of only one coupling constant g means that interaction between quarks and gluons is independent of the quarks flavors.

There is no ordinary definition of quark mass, since quarks have not been observed as asymptotically free states. Below are presented the values of current quark masses, which refer to the masses of the quarks by themselves. They should be distinguished from the constituent

f	u	d	s	c	b	t
m_f, GeV	0.005	0.009	0.175	(1.15 - 1.35)	(4.0 - 4.4)	174

Table 1: *Quark masses. The result is given for the* $\overline{MS}$ *running mass at scale* μ *=1 GeV*

quark masses of a non relativistic quark model which are typically of the order 350 MeV.

From this table we see, that at low energies (∼1 GeV) one can omit effects due to heavy quarks and therefore approximate the full QCD Lagrangian by its light-flavor version. Moreover, because of smallness of light-flavor quark masses, one can consider the QCD Lagrangian in the so-called chiral limit $m_u, m_d, m_s \to 0$ as a good first approximation

$$\mathcal{L}^0_{qcd} = \sum_{l=u,d,s} \bar{q}_l i\not{D}\, q_l - \frac{1}{4}G_{\mu\nu,a}G^{\mu\nu}_a. \tag{2}$$

The Lagrangian $\mathcal{L}^0_{qcd}$ has additional symmetry, called chiral symmetry. In order to see this, let's rewrite the $\mathcal{L}^0_{qcd}$ in terms of left- and right-handed quark fields

$$q_R = \frac{1}{2}(1+\gamma_5)q, \quad q_L = \frac{1}{2}(1-\gamma_5)q.$$

We obtain

$$\mathcal{L}^0_{qcd} = \sum_{l=u,d,s} (\bar{q}_{R,l} i\not{D}\, q_{R,l} + \bar{q}_{L,l} i\not{D}\, q_{L,l}) - \frac{1}{4} G_{\mu\nu,a} G^{\mu\nu}_a. \tag{3}$$

Thus, the $\mathcal{L}^0_{qcd}$ is invariant under the transformations

$$q_R \mapsto U_R q_R, \quad q_L \mapsto U_L q_L,$$

where U_L and U_R are independent unitary 3×3 matrices:

$$U_R = \exp\left(-i \sum_{a=1}^{8} \Theta^R_a \frac{\lambda_a}{2}\right) e^{-i\Theta^R}, \quad U_L = \exp\left(-i \sum_{a=1}^{8} \Theta^L_a \frac{\lambda_a}{2}\right) e^{-i\Theta^L};$$

here the Gell-Mann matrices act in flavor space. Therefore, the Lagrangian $\mathcal{L}^0_{qcd}$ has a classical global $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$ symmetry.

According to Noether's theorem, the consequence of this global symmetry is the existence of conserved currents. Here we will write them down (see [4] for derivations):

$$V^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q, \quad A^{\mu,a} = \bar{q}\gamma^\mu\gamma_5 \frac{\lambda^a}{2} q, \quad V^\mu = \bar{q}\gamma^\mu q, \quad A^\mu = \bar{q}\gamma^\mu\gamma_5 q$$

However, the singlet axial-vector current A_μ is no more conserved after quantization, because of anomalies. Therefore, we are left with the invariance of the $\mathcal{L}^0_{qcd}$ under global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ transformations.

We also introduce charge operators, which are defined as follows:

$$Q^a_V(t) = \int dx\, V^{0,a}(x), \quad Q^a_A(t) = \int dx\, A^{0,a}(x), \quad Q_V(t) = \int dx\, V^0(x).$$

For conserved symmetry currents, these operators are time independent, i.e., they commute with the QCD Hamiltonian H^0_{qcd},

$$[Q^a_V, H^0_{qcd}] = [Q^a_A, H^0_{qcd}] = [Q_V, H^0_{qcd}] = 0$$

They form the Lie algebra of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ group [4]:

$$[Q^a_V, Q^a_V] = i f_{abc} Q^c_V, \quad [Q^a_A, Q^a_A] = i f_{abc} Q^c_V, \quad [Q^a_V, Q^a_A] = i f_{abc} Q^c_A,$$

$$[Q^a_V, Q_V] = [Q^a_A, Q_V] = 0,$$

where f_{abc} are structure constants of SU(3) group. Thus, these charges can be used as generators of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$.

In real world, the chiral symmetry is not exact, because quarks have nonzero masses. Adding the mass term to the Lagrangian of QCD

$$\mathcal{L}_M = -\bar{q}Mq$$

where

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

explicitly breaks the chiral symmetry and the corresponding currents are no more conserved.

Next we want to consider some important consequences of the symmetries in quantum field theory, and particularly in QCD. In quantum field theory, we are interested in objects called Green functions which are vacuum expectation values of time-ordered products. They are of a great importance, because we can calculate any physical observable, if we know them. If theory has some symmetries, then these symmetries put constraints on transformation behavior of Green functions and also relate different Green functions. Such symmetry relations are known as Ward-Takahashi identities. In particular, in QCD one considers the so called chiral Ward identities, which relate the divergence of a Green function containing at least one factor of $V^{\mu,a}$ or $A^{\mu,a}$ to some linear combination of other Green functions. The word *chiral* refers to the underlying $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ group.

It turns out that the set of all chiral Ward identities is encoded as an invariance property of the generating functional of the theory. More precisely, one introduces into the QCD Lagrangian external c-number fields (sources, [5], [6]), which couple to the currents defined above (except to the singlet-axial current):

$$\mathcal{L} = \mathcal{L}^0_{qcd} + \mathcal{L}_{ext} = \mathcal{L}^0_{qcd} + \bar{q}\gamma_\mu(v^\mu + \frac{1}{3}v^\mu_{(s)} + \gamma_5 a^\mu)q - \bar{q}(s - i\gamma_5 p)q. \tag{4}$$

These external fields are color-neutral, Hermitian 3×3 matrices:

$$v^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} v^\mu_a, \quad a^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} a^\mu_a, \quad s = \sum_{a=0}^{8} \lambda_a s_a, \quad p = \sum_{a=0}^{8} \lambda_a p_a.$$

Setting $v^\mu = v^\mu_{(s)} = a^\mu = p = 0$ and $s = \mathrm{diag}(m_u, m_d, m_s)$ $(s = 0)$, we obtain the usual three flavor QCD Lagrangian (QCD Lagrangian in the chiral limit). Then, using the generating functional

$$\begin{aligned} \exp[iZ(v,a,s,p)] &= \langle 0;\mathrm{out}|0;\mathrm{in}\rangle_{v,a,s,p} = \langle 0|T\exp\left[i\int d^4x \mathcal{L}_{ext}(x)\right]|0\rangle \\ &= \langle 0|T\exp\left(i\int d^4x \bar{q}(x)\{\gamma_\mu[v^\mu(x) + \gamma_5 a^\mu(x)] - s(x) + i\gamma_5 p(x)\}q(x)\right)|0\rangle, \end{aligned}$$

$$(5)$$

one is able to obtain any Green function consisting of the time-ordered product of color-neutral, Hermitian quadratic forms by taking functional derivatives with respect to the external fields [4]. Now, in the absence of anomalies, the Ward identities obeyed by the Green functions are equivalent to an invariance of the generating functional under a *local* transformation of the external fields [7].

Requirement of the Lagrangian $\mathcal{L}$ to be Hermitian and invariant under parity transformation (P) and charge conjugation (C) leads to the following transformations of the external fields under P and C operations:

$$v^\mu \overset{P}{\mapsto} v_\mu, \quad v^\mu_{(s)} \overset{P}{\mapsto} v^{(s)}_\mu, \quad a^\mu \overset{P}{\mapsto} -a_\mu, \quad s \overset{P}{\mapsto} s, \quad p \overset{P}{\mapsto} -p. \tag{6}$$

$$v_\mu \overset{C}{\to} -v^T_\mu, \quad v^{(s)}_\mu \overset{C}{\to} -v^{(s)T}_\mu, \quad a_\mu \overset{C}{\to} a^T_\mu, \quad s, p \overset{C}{\to} s^T, p^T, \tag{7}$$

The Lagrangian $\mathcal{L}$ can be rewritten in terms of the left- and right-handed quark fields:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^0_{qcd} + \bar{q}_L\gamma^\mu\left(l_\mu + \frac{1}{3}v^{(s)}_\mu\right)q_L + \bar{q}_R\gamma^\mu\left(r_\mu + \frac{1}{3}v^{(s)}_\mu\right)q_R \\ &\quad -\bar{q}_R(s+ip)q_L - \bar{q}_L(s-ip)q_R, \end{aligned} \tag{8}$$

where

$$r_\mu = v_\mu + a_\mu, \quad l_\mu = v_\mu - a_\mu.$$

Equation (8) is invariant under the following *local* transformations of the quark fields and external sources [4]:

$$\begin{aligned} q_R &\mapsto \exp\left(-i\frac{\Theta(x)}{3}\right)R(x)q_R, \\ q_L &\mapsto \exp\left(-i\frac{\Theta(x)}{3}\right)L(x)q_L, \\ r_\mu &\mapsto Rr_\mu R^\dagger + iR\partial_\mu R^\dagger, \\ l_\mu &\mapsto Ll_\mu L^\dagger + iL\partial_\mu L^\dagger, \\ v^{(s)}_\mu &\mapsto v^{(s)}_\mu - \partial_\mu\Theta, \\ s+ip &\mapsto R(s+ip)L^\dagger, \\ s-ip &\mapsto L(s-ip)R^\dagger, \end{aligned} \tag{9}$$

where $R(x)$ and $L(x)$ are independent space-time-dependent SU(3) matrices. Thus, it is possible to make the Lagrangian $\mathcal{L}$ to be invariant under local $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R\times\mathrm{U}(1)_V$ transformations.

The point is that one can consistently and systematically approximate the generating functional $Z(v,a,s,p)$ of QCD at low energies by generating functional of an effective field theory with Lagrangian, that can always be brought to a manifestly locally chiral invariant form by adding total derivatives and performing meson field redefinitions [7]. In particular, this will allow one to study the low-energy behavior of the Green functions of QCD, which at the same time will represent a solution of the Ward identities. The Chiral perturbation theory (ChPT) is an example of such an effective theory.

2.2 ChPT as an effective field theory

At low energies, quarks due to confinement are bound (together with gluons) into particles called hadrons. Proton, neutron, pion are examples of hadrons. We are interested to describe interactions between them. We cannot use underlying QCD theory because there is no way to use it in perturbative manner. Therefore, the question arises whether it is possible to construct such an effective field theory which would replace QCD at low energies and at same time would give a sensible theoretical predictions. It appears that it is possible. Namely, we know the symmetry properties of strong interactions; therefore, we can write an effective field theory in terms of the hadronic asymptotic states, and parametrize the unknown dynamical information in a few coupling. However, we cannot simply compute the effective Lagrangian directly from the original QCD Lagrangian. The connection between the original and effective theories is non-perturbative.

The theoretical basis for construction of such effective field theories was provided in Ref. [8] as a "theorem" (conjecture), which states that *perturbative description in terms of the most general effective Lagrangian containing all possible terms compatible with assumed symmetry principles yields the most general S matrix consistent with the fundamental principles of quantum field theory and the assumed symmetry principles.* The corresponding effective Lagrangian will contain an infinite number of terms with an infinite number of free parameters.

Chiral perturbation theory (ChPT) provides a systematic method for discussing the consequences of the global flavor symmetries of QCD at low energies by means of an effective field theory.At quite low energies, the corresponding Lagrangian is expressed in terms of the lightest hadrons states; these are members of pseudoscalar octet ($\pi^+, \pi^-, \pi^0, \eta, K^+, K^-, K^0$ and $\bar{\mathrm{K}}^0$). Such effective field theory is called the ChPT for mesons. We note that it is also possible to construct the ChPT for baryons (like protons and neutrons), but it is beyond the scope of this thesis.

Before proceeding further, we have to consider one important property of QCD, which is tightly connected to the construction of the ChPT for mesons. Namely, there is experimental evidence that chiral symmetry of QCD is *spontaneously* broken. A continuous symmetry is said to be spontaneously broken or hidden, if the ground state of the system is no longer invariant under the full symmetry group of the Hamiltonian.

Previously, we have seen that the light-flavor QCD Lagrangian possess an $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ symmetry. This chiral symmetry is however not seen in the hadronic spectrum. According to experiment, degenerate multiplets with opposite (negative) parity do not exist. In contrast, hadrons can be nicely classified in SU(3) representations. The explanation is as follows.

As it was shown in Ref. [9], the ground state is necessarily invariant under $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$ transformations, that is the charges Q_V^a and Q_V annihilate the ground state (vacuum):

$$Q_V^a|0\rangle = Q_V|0\rangle = 0.$$

According to Coleman's theorem [10], if the vacuum is invariant under $\mathrm{SU}(3) \times \mathrm{U}(1)_V$, then so is the Hamiltonian (but not vice versa). This further implies that the physical states of the spectrum of the QCD Hamiltonian H^0_{qcd} can be organized according to irreducible representations of $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$. The index V indicates that the generators transform with a positive sign under parity. The $\mathrm{U}(1)_V$ symmetry results in baryon number conservation and leads to a classification of hadrons into mesons (B = 0) and baryons (B = 1). Then, since the parity doubling is not observed for the low-lying states, one assumes that the Q_A^a do *not* annihilate the ground state:

$$Q_A^a|0\rangle \neq 0.$$

Thus, the $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry *spontaneously* breaks down to $\mathrm{SU}(3)_V$.

In accordance with Goldstone's theorem [11, 12], to each axial generator Q_A^a, which does not annihilate the ground state, corresponds a massless Goldstone boson field $\phi^a(x)$ with spin 0, whose symmetry properties are closely connected to the generator in question. In particular, the Goldstone bosons are pseudoscalars, which means that they transform under parity as

$$\phi^a(t, \vec{x}) \overset{P}{\mapsto} -\phi^a(t, -\vec{x}) \tag{10}$$

Also, they transform under the subgroup $\mathrm{SU}(3)_V$ as an octet:

$$[Q_V^a, \phi^b(x)] = i f_{abc}\phi^c(x)$$

Since there are eight broken axial generators of the chiral group, Q_A^a, there should be eight pseudoscalar Goldstone states, which we can identify with the eight lightest hadronic states (π, η, K). The non-vanishing masses of the light pseudoscalars in the real world are related to the explicit symmetry breaking in QCD due to the light quark masses.

Additionally, we would like to mention theoretical conditions for a spontaneous chiral symmetry breaking in QCD [4]. Firstly, a non-vanishing scalar quark condensate, which is the quantity $\langle 0|\bar{q}q|0\rangle$ is a sufficient but not a necessary condition for a spontaneous chiral symmetry breakdown in QCD:

$$\langle 0|\bar{q}q|0\rangle \neq 0$$

Secondly, considering the nonzero matrix element of the axial-vector current between the vacuum and massless one particle states $|\phi^b\rangle$, which because of Lorentz covariance can be written as

$$\langle 0|A_\mu^a(0)|\phi^b(p)\rangle = ip_\mu F_0 \delta^{ab},$$

one obtains that nonzero value of F_0 (this constant will be introduced again later) is a necessary and sufficient criterion for spontaneous chiral symmetry breaking.

Returning to ChPT, the basic assumption of ChPT is that the chiral limit constitutes a realistic starting point for a systematic expansion in chiral symmetry breaking interactions. The Goldstone nature of the pseudoscalar mesons implies strong constraints on their interactions. Here we mention some essential properties of interactions between Goldstone bosons [13]:

- The Goldstone boson fields are derivatively coupled. Thus only gradients of fields appear in the Lagrangian.
- The effective Lagrangian describes a theory of *weakly* interacting Goldstone bosons at low energies. The Goldstone boson couplings are proportional to their momentum, and so vanish for low-momentum Goldstone bosons.
- The Goldstone boson Lagrangian is non-linear in the Goldstone boson fields. It describes the dynamics of fields constrained to live on the vacuum manifold, which is generically curved.

The general formalism for effective Lagrangians for spontaneously broken symmetries was worked out by Callan, Coleman, Wess and Zumino [14, 15] and is known as CCWZ formalism. Following this formalism and applying it to QCD, the Goldstone fields are collected in a unitary matrix field $U(\phi)$ transforming as

$$U(\phi) \mapsto RU(\phi)L^\dagger, \qquad L \in \mathrm{SU}(3)_L, \quad R \in \mathrm{SU}(3)_R \tag{11}$$

under chiral rotations $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$. There are different parameterizations of $U(\phi)$ corresponding to different choices of coordinates for the chiral coset space $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R/\mathrm{SU}(3)_V$. For convenience one chooses the matrix $U(x) \equiv U(\phi(x))$ to be the SU(3) matrix:

$$U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right),$$

where

$$\phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \tag{12}$$

Now, using the transformation law $U \mapsto RUL^\dagger$ one can construct the most general, chirally invariant, effective Lagrangian; with the minimal number of derivatives it reads

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4} \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right), \tag{13}$$

where $F_0 \approx 93$ MeV is a free parameter known as pion decay constant, which is related to the pion decay $\pi^+ \to \mu^+\nu_\mu$. Expansion of U in a power series in the meson fields gives the right kinetic term:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\partial_\mu\phi_a(x)\partial^\mu\phi_a(x) + \mathcal{L}_{int},$$

where the interaction Lagrangian $\mathcal{L}_{int}$ starts with interaction terms containing at least four Goldstone bosons. If we perform the substitution $\phi_a(t,\vec{x}) \mapsto -\phi^a(t,\vec{x})$ for the Goldstone boson fields, or equivalently, $U(t,\vec{x}) \mapsto U^\dagger(t,\vec{x})$, in the $\mathcal{L}_{\text{eff}}$, then $\mathcal{L}_{\text{eff}}$ doesn't change. It means that the $\mathcal{L}_{\text{eff}}$ contains in it's expansion only terms with even number of Goldstone boson fields. In this case the Lagrangian $\mathcal{L}_{\text{eff}}$ is of so-called even *intrinsic* parity. There also exists odd intrinsic parity sector of the mesonic ChPT [16, 17], but we are interested in even intrinsic parity sector only.

However, as we saw, in real world the symmetry $\mathrm{SU(3)}_L \times \mathrm{SU(3)}_R$ is not perfect, because the QCD Lagrangian contains quark mass term

$$\mathcal{L}_M = -\bar{q}Mq = -\bar{q}_R M q_L - \bar{q}_L M^\dagger q_R.$$

In order to incorporate the consequences of this fact into the effective-Lagrangian framework, one makes use of the following argument [18]: even though M is in reality just a constant matrix and does not transform along with the quark fields, $\mathcal{L}_M$ *would be* invariant if M transformed as

$$M \mapsto RML^\dagger.$$

One then constructs the most general Lagrangian, which is invariant under simultaneous transformations $U \mapsto RUL^\dagger, \quad M \mapsto RML^\dagger$. At lowest order in M one obtains

$$\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2}\mathrm{Tr}(MU^\dagger + UM^\dagger),$$

where the subscript s.b. refers to symmetry breaking. Here B_0 is the new constant, which can be related to the chiral quark condensate [4] as

$$3F_0^2 B_0 = -\langle 0|\bar{q}q|0\rangle.$$

Substituting the quark-mass matrix M and expanding the $\mathcal{L}_{\text{s.b.}}$ in power series in the meson fields, one reads off the masses of the Goldstone bosons, to lowest order in the quark masses,

$$M_\pi^2 = 2B_0 m,$$

$$
\begin{aligned}
M_K^2 &= B_0(m+m_s), \\
M_\eta^2 &= \frac{2}{3}B_0\left(m+2m_s\right),
\end{aligned} \tag{14}
$$

where for the sake of simplicity $m_u = m_d = m$ is set. Using the relation $B_0 = -\langle 0|\bar{q}q|0\rangle/(3F_0^2)$, we see that quadratic masses of the Goldstone bosons linearly depend on the quark condensate and the quark masses. The latter result is supported by the analysis of the data on $K^+ \to \pi^+\pi^-e^+\nu_e$ [19][20], which means that the quark condensate really characterizes spontaneous chiral symmetry breaking in QCD.

2.3 Effective Lagrangians in ChPT

As we have seen, the effective chiral Lagrangian contains infinite number of terms, which have the same symmetry properties as underlying theory, i.e. QCD. In the ChPT, the most general chiral Lagrangian describing the dynamics of the Goldstone bosons is organized as an infinite sum of terms with an increasing number of derivatives and quark mass terms,

$$
\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots, \tag{15}
$$

where the subscripts refer to the order in a momentum and quark mass expansion. According to formulas (14) and on-shell condition $p^2 = M^2$, for consistency, one should count one quark-mass term as being of the same order as two derivatives:

$$
m_q \sim O(p^2).
$$

Therefore, the index 2 denotes either two derivatives or one quark mass term. Consequently, the $\mathcal{L}_2$ contains terms of so-called chiral order $O(p^2)$. Analogously, $\mathcal{L}_4$ denotes terms of chiral order $O(p^4)$ with corresponding numbers of derivatives and quark mass terms etc.

Since we are interested in making sensible predictions using this effective Lagrangian, we need some rule, which would tell us what diagrams one has to take into account when calculating given physical matrix element with defined accuracy. Such a rule was given in Ref. [8] and is known as Weinberg's power counting scheme (or argument). It analyzes the behavior of a given diagram under a linear rescaling of all the *external* momenta, and a quadratic rescaling of the light quark masses (quadratic Goldstone boson masses):

$$
p_i \mapsto tp_i, \quad m_q \mapsto t^2 m_q \quad (M^2 \mapsto t^2 M^2).
$$

Let $\mathcal{A}(p_i, m_q)$ to be the amplitude of a given diagram. After rescaling it takes the form

$$
\mathcal{A}(tp_i, t^2 m_q) = t^{\mathcal{D}} \mathcal{A}(p_i, m_q).
$$

The $\mathcal{D}$ is a number called the chiral dimension of a given diagram and is equal to

$$\mathcal{D} = 2 + 2L + \sum_{n=1}^{\infty} 2(n-1)V_{2n}, \tag{16}$$

where V_{2n} denotes the number of vertices originating from $\mathcal{L}_{2n}$ and L is a number of independent loops [4]. Going to small enough momenta and masses, such that the t changes in the range $0 < t < 1$, means that diagrams with small $\mathcal{D}$, such as $\mathcal{D} = 2, 4$, should dominate. Moreover, the given diagram with chiral dimension $\mathcal{D}$ is of chiral order $O(p^{\mathcal{D}})$. Thus, we conclude, that in order to calculate a physical matrix element with a given finite accuracy, it is sufficient to consider only finite number of diagrams.

Calculating loop graphs, we might expect, that a given amplitude is proportional to some power of the expansion parameter p/Λ_χ, where Λ_χ is some typical hadronic scale. The loop expansion suggests

$$\Lambda_\chi \sim 4\pi F_0 \approx 1.2 \text{ GeV}$$

as a natural scale of the chiral expansion [21]. This parameter is large enough that one can apply chiral Lagrangians to low energy processes involving pions and kaons. If it were just F_0, then chiral effective Lagrangians would not be useful even for pions, since their mass is approximately $M_\pi \approx 140$ MeV, while F_0 is 93 MeV. Restricting the domain of applicability of ChPT to momenta $|p| = O(M_K)$, where $M_K \approx 500$ MeV is kaon mass, the expansion parameter is expected to be

$$\frac{M_K^2}{16\pi^2 F_0^2} = 0.18.$$

There is also improved estimate of Λ_χ [22, 23]:

$$\Lambda_\chi \sim \frac{4\pi F_0}{\sqrt{N_f}},$$

where N_f is the number of light flavors (N_f=2, 3). It stems from the fact, that the greater N_f is, the more number of mesons can run in loops. Therefore, one would expect considerably better convergence of the chiral expansion in the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ framework, because in this case $N_f = 2$ and $|p| = O(M_\pi)$. Finally, one should also mention that the so-called chiral logarithms emerge from the loops, so the convergence of the perturbative expansion is in fact slower than can be concurred from the above "rule of the thumb".

Next, in connection with mentioned at the end of subsection 1.1, we want to promote the global symmetry of the effective Lagrangian $\mathcal{L}_{\text{eff}}$ to a local one. Then, using this new locally chiral invariant Lagrangian $\mathcal{L}_{\text{eff}}$, we can approximate the generating functional of QCD at low energies by the generating functional, obtained with help of the effective field theory:

$$e^{iZ_{qcd}[v,a,s,p]} \approx e^{iZ[v,a,s,p]} = \int [dU(\phi)] e^{i\int d^4x \mathcal{L}_{\text{eff}}}, \tag{17}$$

where $[dU(\phi)]$ denotes the measure of the functional integral.

In order to construct the effective chiral Lagrangian for a local $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry, one introduces the *same* external fields v, a, s and p as in QCD, and defines the covariant derivative $d_\mu A$ for any object transforming as $A \mapsto RAL^\dagger$:

$$d_\mu A = \partial_\mu A - ir_\mu A + iAl_\mu, \quad d_\mu A \mapsto R(d_\mu A)L^\dagger.$$

It transforms in the same way as the object A. Also, the following combinations are defined:

$$\chi = 2B_0(s + ip),$$

$$R_{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \quad L_{\mu\nu} = \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu],$$

where $R_{\mu\nu}$ and $L_{\mu\nu}$ are the field strength tensors associated with the r_μ and l_μ correspondingly. Introduced expressions can be used as the building blocks for construction of the locally chiral

element	G	C	P
U	$RUL^\dagger$	U^T	$U^\dagger$
$d_{\lambda_1} \cdots d_{\lambda_n} U$	$Rd_{\lambda_1} \cdots d_{\lambda_n} UL^\dagger$	$(d_{\lambda_1} \cdots d_{\lambda_n} U)^T$	$(d^{\lambda_1} \cdots d^{\lambda_n} U)^\dagger$
χ	$R\chi L^\dagger$	χ^T	$\chi^\dagger$
$d_{\lambda_1} \cdots d_{\lambda_n} \chi$	$Rd_{\lambda_1} \cdots d_{\lambda_n} \chi L^\dagger$	$(d_{\lambda_1} \cdots d_{\lambda_n} \chi)^T$	$(d^{\lambda_1} \cdots d^{\lambda_n} \chi)^\dagger$
r_μ	$Rr_\mu R^\dagger + iR\partial_\mu R^\dagger$	$-l_\mu^T$	l^μ
l_μ	$Ll_\mu L^\dagger + iL\partial_\mu L^\dagger$	$-r_\mu^T$	r^μ
$R_{\mu\nu}$	$RR_{\mu\nu}R^\dagger$	$-(L_{\mu\nu})^T$	$L^{\mu\nu}$
$L_{\mu\nu}$	$LL_{\mu\nu}L^\dagger$	$-(R_{\mu\nu})^T$	$R^{\mu\nu}$

Table 2: *Transformation properties of the building blocks under the group (G), charge conjugation (C), and parity (P). The expressions for adjoint matrices are obtained by taking the Hermitian conjugate of each entry.*

invariant effective Lagrangian. In Table 2 are presented the transformation properties of all building blocks under the group (G), charge conjugation (C), and parity (P).

In the chiral counting scheme of ChPT the elements for consistency should be counted as:

$$U = O(1),\ D_\mu U = O(p),\ r_\mu, l_\mu = O(p),\ R_{\mu\nu}, L_{\mu\nu} = O(p^2),\ \chi = O(p^2). \tag{18}$$

and any additional covariant derivative counts as $O(p)$. Using this counting rule and Table 2 for the building blocks, we can construct the most general, Lorentz, C, P and locally-invariant effective Lagrangian at lowest chiral order $O(p^2)$ [4, 5, 6]; it is of the form

$$\mathcal{L}_2 = \frac{F_0^2}{4}\mathrm{Tr}[d_\mu U(d^\mu U)^\dagger] + \frac{F_0^2}{4}\mathrm{Tr}[\chi U^\dagger + U\chi^\dagger]. \tag{19}$$

The Lagrangian $\mathcal{L}_2$ can be written as

$$\mathcal{L}_2 = \frac{F_0^2}{4}\langle d_\mu U d^\mu U^\dagger + \chi U^\dagger + U\chi^\dagger\rangle,$$

where it assumed that $\langle \ldots \rangle \equiv \mathrm{Tr}[\ldots]$ and $d^\mu U^\dagger \equiv (d^\mu U)^\dagger$. Substituting the scalar density s expansion around the quark-mass matrix

$$s = M + \ldots$$

in $\mathcal{L}_2$, we obtain the same relations for meson masses (14), which justify the chiral counting rule (18).

In the same way it is possible to construct the most general Lagrangian at next-to-leading order, i.e. at chiral order $O(p^4)$ [5]. It reads

$$\begin{aligned}
\mathcal{L}_4 \;=\;& L_1\langle d_\mu U^\dagger d^\mu U\rangle^2 + L_2\langle d_\mu U^\dagger d_\nu U\rangle\langle d^\mu U^\dagger d^\nu U\rangle \\
&+L_3\langle d_\mu U^\dagger d^\mu U d_\nu U^\dagger d^\nu U\rangle + L_4\langle d_\mu U^\dagger d^\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
&+L_5\langle d_\mu U^\dagger d^\mu U(\chi^\dagger U + U^\dagger\chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 + L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 \\
&+L_8\langle \chi^\dagger U\chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle - iL_9\langle R^{\mu\nu} d_\mu U d_\nu U^\dagger + L^{\mu\nu} d_\mu U^\dagger d_\nu U\rangle \\
&+L_{10}\langle U^\dagger R^{\mu\nu} U L_{\mu\nu}\rangle + H_1\langle R_{\mu\nu}R^{\mu\nu} + L_{\mu\nu}L^{\mu\nu}\rangle + H_2\langle \chi^\dagger\chi\rangle \;. \qquad (20)
\end{aligned}$$

and satisfies local chiral invariance, Lorentz invariance, P and C. We see that while at leading order one needs two constants F_0, B_0 to determine the low-energy behavior of the Green functions, at next-to-leading order it is necessary 10 additional low-energy coupling constants $L_1, \ldots, L_{10}$ (the terms H_1, H_2 are of no physical relevance, since they contain only external fields).

As one can see, the Lagrangian $\mathcal{L}_4$ contains terms which are not presented in $\mathcal{L}_2$. This is the general feature of effective field theories, which are non-renormalizable in a usual sense like QED or QCD. However, ChPT Lagrangian $\mathcal{L}_{\text{eff}}$ is the most general chiral invariant Lagrangian, and since the divergences can be absorbed by local counterterms that exhibit the same symmetries as the initial Lagrangian [24], it automatically includes all terms needed for renormalization to every order in the loop expansion.

Consider one–loop diagrams generated by the $\mathcal{L}_2$. They are of order $O(p^4)$, since according to Eq. (16), $L = 1$ for $\mathcal{D} = 4$. Using dimensional regularization, which preserves symmetries of theory, in particular chiral symmetry, one finds that the counter terms necessary to absorb divergences produced by the one–loop diagrams, have the structure of the terms presented in the next-to-leading order Lagrangian $\mathcal{L}_4$. Therefore the one-loop divergences can be eliminated by an appropriate renormalization of the low-energy constants L_i and H_i. Later, in connection with our thesis problem, we will consider this more precisely.

3 Virtual photons in ChPT

3.1 Effective Lagrangian at $O(p^4)$

In real world, the pseudoscalar mesons not only have masses, but some of them, for example π^+, K^+ are also electrically charged particles. Therefore it is necessary to include electromagnetic interactions between them in ChPT framework. To that end, consider the Lagrangian term, which is responsible for the interaction of quarks to the electromagnetic field:

$$\mathcal{L}_{\text{em}} = -\bar{q}QA_\mu\gamma^\mu q = -\bar{q}_R QA_\mu\gamma^\mu q_R - \bar{q}_L QA_\mu\gamma^\mu q_L,$$

where A_μ is the electromagnetic field potential and Q is the quark charge matrix

$$Q = \frac{e}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

If we introduce the so called spurion fields $Q_R(x)$, $Q_L(x)$ and rewrite $\mathcal{L}_{\text{em}}$ as follows

$$\mathcal{L}_{\text{em}} = -\bar{q}_R Q_R A_\mu\gamma^\mu q_R - \bar{q}_L Q_L A_\mu\gamma^\mu q_L,$$

then $\mathcal{L}_{\text{em}}$ will be locally chiral invariant, if the spurions transform under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ as

$$Q_R(x) \mapsto RQ_R(x)R^\dagger, \quad Q_L(x) \mapsto LQ_L(x)L^\dagger. \tag{21}$$

Additionally, it is possible the following modification of the covariant derivative $d_\mu U$:

$$d_\mu U = \partial_\mu U - iR_\mu U + iUL_\mu,$$

with

$$R_\mu = v_\mu + a_\mu + A_\mu Q_R, \quad L_\mu = v_\mu - a_\mu + A_\mu Q_L.$$

In order to save the previously introduced power counting scheme one puts

$$Q_R,\, Q_L = O(p), \quad A_\mu = O(1).$$

Using the spurions $Q_R(x)$, $Q_L(x)$ as additional building blocks and the counting rule for them, one can construct the most general Lagrangian, which includes electromagnetic interactions and which is consistent with the chiral symmetry, P and C invariance. One then sets the spurion fields to the constant charge matrix Q:

$$Q_R(x) = Q_L(x) = Q.$$

At leading order, i.e. at chiral order $O(p^2)$, it reads [26, 27]

$$\begin{aligned}\mathcal{L}_2^{(Q)} &= \frac{F_0^2}{4}\langle d^\mu U^+ d_\mu U + \chi^+ U + U^+\chi\rangle - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &\quad -\frac{1}{2a}(\partial^\mu A_\mu)^2 + C\langle Q_R U Q_L U^+\rangle, \end{aligned} \tag{22}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor, a is the gauge fixing parameter and C is the new constant, which determines the purely electromagnetic part of the masses of the charged pions and kaons in the chiral limit,

$$M_{\pi^\pm}^2 = M_{K^\pm}^2 = 2e^2\frac{C}{F_0^2} + O(m_q)$$

The Lagrangian (22) generates one-loop graphs consisting of meson and photon lines. They are of order $O(p^4)$ and contain divergences, which should be absorbed by adding tree graphs, evaluated with the next-to-leading order Lagrangian $\mathcal{L}_4^{(Q)}$. Consider loop expansion from the point of view of path integral formulation of quantum field theory. The generating functional of ChPT (17) reads, up to and including terms of order $O(p^4)$

$$e^{iZ[v,a,s,p]} = \int [dU][dA_\mu] e^{i\int d^4x\left\{\mathcal{L}_2^{(Q)}+\mathcal{L}_4^{(Q)}\right\}}, \tag{23}$$

where $[dA_\mu]$ means the path integral measure for electromagnetic field. One should calculate $Z[v,a,s,p]$ at one-loop level. To this purpose, we note that the classical field theory associated with a given Lagrangian is equivalent to the set of tree graphs of the corresponding quantum field theory. Thus if we use the classical field equations to evaluate $Z[v,a,s,p]$, then $Z[v,a,s,p]$ generates Green functions at tree approximation (leading order) [29, 30].

Since the vertices of the Lagrangian $\mathcal{L}_4^{(Q)}$ only occur in tree graphs, the contribution from $\mathcal{L}_4^{(Q)}$ to the generating functional can be calculated by evaluating the action $\int dx\, \mathcal{L}_4^{(Q)}$ at the classical solution of the equations of motion. Therefore the most general Lagrangian at $O(p^4)$ can be simplified with the help of the equations of motion.

The next-to-leading order Lagrangian in the presence of virtual photons was constructed in Ref. [27]. Additional building blocks with their transformation properties are presented in Table 3. The quantities $c_\mu^R Q_R$, $c_\mu^L Q_L$ are defined as

element	G	C	P
Q_R	$RQ_RR^\dagger$	Q_L^T	Q_L
$c_\mu^R Q_R$	$Rc_\mu^R Q_R R^\dagger$	$(c_\mu^L Q_L)^T$	$c^{L\mu}Q_L$
Q_L	$LQ_LL^\dagger$	Q_R^T	Q_R
$c_\mu^L Q_L$	$Lc_\mu^L Q_L L^\dagger$	$(c_\mu^R Q_R)^T$	$c^{R\mu}Q_R$

Table 3: *Transformation properties of the additional building blocks under the group (G), charge conjugation (C), and parity (P).*

$$c_\mu^I Q_I = \partial_\mu Q_I - i[I_\mu, Q_I], \qquad\qquad I = R, L\ .$$

They transform under $\mathrm{SU}(3)_R \times \mathrm{SU}(3)_L$ in the same way as Q_R and Q_L. Since one further sets $Q_R = Q_L = Q = \mathrm{const}$, so $c_\mu^I Q = -i[I_\mu, Q]$. The Lagrangian $\mathcal{L}_4^{(Q)}$ has the following form:

$$\begin{aligned}
\bar{\mathcal{L}}_4^{(Q)} \;=\;& \bar{\mathcal{L}}_{p^4} + F_0^2\{K_1\langle d^\mu U^+ d_\mu U\rangle\langle Q^2\rangle + K_2\langle d^\mu U^+ d_\mu U\rangle\langle QUQU^+\rangle \\
&+K_3(\langle d^\mu U^+ QU\rangle\langle d_\mu U^+ QU\rangle + \langle d^\mu UQU^+\rangle\langle d_\mu UQU^+\rangle) \\
&+K_4\langle d^\mu U^+ QU\rangle\langle d_\mu UQU^+\rangle + K_5\langle(d^\mu U^+ d_\mu U + d^\mu U d_\mu U^+)Q^2\rangle \\
&+K_6\langle d^\mu U^+ d_\mu UQU^+ QU + d^\mu U d_\mu U^+ QUQU^+\rangle \\
&+K_7\langle\chi^+ U + U^+\chi\rangle\langle Q^2\rangle + K_8\langle\chi^+ U + U^+\chi\rangle\langle QUQU^+\rangle \\
&+K_9\langle(\chi U^+ + U\chi^+ + \chi^+ U + U^+\chi)Q^2\rangle \\
&+K_{10}\langle(\chi U^+ + U\chi^+)QUQU^+ + (\chi^+ U + U^+\chi)QU^+ QU\rangle \\
&+K_{11}\langle(\chi U^+ - U\chi^+)QUQU^+ + (\chi^+ U - U^+\chi)QU^+ QU\rangle \\
&+K_{12}\langle d_\mu U^+[c_R^\mu Q, Q]U + d_\mu U[c_L^\mu Q, Q]U^+\rangle \\
&+K_{13}\langle c_R^\mu QU c_{L\mu} QU^+\rangle + K_{14}\langle c_R^\mu Q c_{R\mu} Q + c_L^\mu Q c_{L\mu} Q\rangle\} \\
&+F_0^4\{K_{15}\langle QUQU^+\rangle^2 + K_{16}\langle QUQU^+\rangle\langle Q^2\rangle + K_{17}\langle Q^2\rangle^2\},
\end{aligned} \tag{24}$$

where it is supposed, that $U = \bar{U}, A_\mu = \bar{A}_\mu$ are classical solutions, which are determined by the equations of motion,

$$\begin{aligned}
d_\mu d^\mu \bar{U}\bar{U}^+ - \bar{U} d_\mu d^\mu \bar{U}^+ + \bar{U}\chi^+ - \chi\bar{U}^+ - \frac{1}{3}\langle\bar{U}\chi^+ - \chi\bar{U}^+\rangle& \\
+\frac{4C}{F_0^2}\left(\bar{U}Q\bar{U}^+ Q - Q\bar{U}Q\bar{U}^+\right) &= 0, \\
\left[g_{\mu\nu}\partial^2 - \left(1 - \frac{1}{a}\right)\partial_\mu\partial_\nu\right]\bar{A}^\nu + \frac{iF_0^2}{2}\langle d_\mu\bar{U}[\bar{U}^+, Q]\rangle = 0.&
\end{aligned} \tag{25}$$

The Lagrangian $\bar{\mathcal{L}}_{p^4}$ comes from the strong sector and is given by Eq. (20).

3.2 Generating functional at one-loop

The generating functional of Eq. (23) becomes

$$e^{iZ[v,a,s,p]} = e^{i\int d^4x\bar{\mathcal{L}}_4^{(Q)}} \int [dU][dA_\mu] e^{i\int d^4x \mathcal{L}_2^{(Q)}}$$

To evaluate the one-loop graphs produced by the Lagrangian $\mathcal{L}_2^{(Q)}$, we expand the fields $U(x), A_\mu(x)$ in the neighborhood of the classical solutions $\bar{U}$, $\bar{A}_\mu$ [32]:

$$\begin{aligned} U &= u e^{i\xi/F_0} u = u\left(\mathbf{1} + i\frac{\xi}{F_0} - \frac{1}{2}\frac{\xi^2}{F_0^2} + \cdots\right) u \\ &= \bar{U} + \frac{i}{F_0} u\xi u - \frac{1}{2F_0^2} u\xi^2 u + \ldots \\ A_\mu &= \bar{A}_\mu + \epsilon_\mu, \end{aligned} \tag{26}$$

where $\bar{U} = u^2$ and ξ is a traceless hermitian matrix, $\xi = \sum_a \xi^a \lambda^a$. Then we substitute this expansion in the action $S = \int dx \mathcal{L}_2^{(Q)}$ and keep only terms, quadratic in the fluctuations ξ, ϵ_μ. As a result we obtain [27, 31]

$$S = \int dx \bar{\mathcal{L}}_2^{(Q)} - \frac{1}{2}\int dx \eta_A D^{AB} \eta_B,$$

where the fluctuations are collected in a new flavour space elements $\eta_A = (\xi_a, \epsilon_\mu) = (\xi_1, \ldots, \xi_8, \epsilon_0, \ldots, \epsilon_3)$ and matrix D is the differential operator defined as follows:

$$D = D_0 + \delta, \tag{27}$$

$$D_0 = \begin{pmatrix} \partial^2 \delta^{ab} & 0 \\ 0 & -\partial^2 g^{\sigma\rho} + \left(1 - \frac{1}{a}\right)\partial^\sigma \partial^\rho \end{pmatrix}, \tag{28}$$

$$\delta(x) = \{Y_\mu, \partial^\mu\} + Y_\mu Y^\mu + \Lambda, \tag{29}$$

with

$$Y_\mu(x) = \begin{pmatrix} \Gamma_\mu^{ab} & X_\mu^{a\rho} \\ X_\mu^{\sigma b} & 0 \end{pmatrix}, \quad \Lambda(x) = \begin{pmatrix} \sigma^{ab} & -\frac{1}{2}\gamma^{a\rho} \\ -\frac{1}{2}\gamma^{\sigma b} & -\rho g^{\sigma\rho} \end{pmatrix}. \tag{30}$$

The elements of these matrices are given by the expressions:

$$\begin{aligned} \Gamma_\mu^{ab} &= -\frac{1}{2}\langle [\lambda^a, \lambda^b]\Gamma_\mu \rangle, \\ X_\mu^{a\rho} &= -X_\mu^{\rho a} = X^a \delta_\mu^\rho, \quad X^a = -\frac{1}{4}\langle H_L \lambda^a \rangle, \\ \sigma^{ab} &= \frac{1}{2}\langle [\Delta_\mu, \lambda^a][\Delta^\mu, \lambda^b] \rangle + \frac{1}{4}\langle \{\lambda^a, \lambda^b\}\sigma \rangle - \frac{F_0^2}{4}\langle H_L \lambda^a \rangle \langle H_L \lambda^b \rangle \end{aligned}$$

$$
\begin{aligned}
& & -\frac{C}{8F_0^2}\left\{\langle[H_R+H_L,\lambda^a][H_R-H_L,\lambda^b]+a\leftrightarrow b\rangle\right\}, \\
\gamma^{a\rho} &=& \gamma^{\rho a}=F_0\langle\left([H_R,\Delta^\rho]+\frac{1}{2}D^\rho H_L\right)\lambda^a\rangle, \\
\rho &=& \frac{3}{8}F_0^2\langle H_L^2\rangle,
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
D_\mu H_L &=& \partial_\mu H_L+[\Gamma_\mu,H_L] \\
\Gamma_\mu &=& \frac{1}{2}[u^+,\partial_\mu u]-\frac{1}{2}iu^+\bar{R}_\mu u-\frac{1}{2}u\bar{L}_\mu u^+, \\
\Delta_\mu &=& \frac{1}{2}u^+d_\mu\bar{U}u^+=-\frac{1}{2}ud_\mu\bar{U}^+u, \\
H_R &=& u^+Q_Ru+uQ_Lu^+, \\
H_L &=& u^+Q_Ru-uQ_Lu^+, \\
\sigma &=& \frac{1}{2}(u^+\chi u^++u\chi^+u).
\end{aligned}
\tag{32}
$$

The generating functional thus takes the form

$$
e^{iZ[v,a,s,p]}=e^{i\int dx\left\{\bar{\mathcal{L}}_2^{(Q)}+\bar{\mathcal{L}}_4^{(Q)}\right\}}\int[d\xi_a][d\epsilon_\mu]e^{-\frac{i}{2}\int dx\eta_A D^{AB}\eta_B}.
$$

The remaining path integral over fluctuations reduces to a Gaussian integral and we finally obtain $Z[v,a,s,p]$ at one-loop:

$$
Z[v,a,s,p]=\int dx\bar{\mathcal{L}}_2^{(Q)}+\int dx\bar{\mathcal{L}}_4^{(Q)}+\frac{i}{2}\ln\det D,
\tag{33}
$$

where all quantities are to be evaluated at the classical solutions $\bar{U}(x),\bar{A}_\mu(x)$. The determinant of the operator D requires renormalization, since it contains divergences of one-loop graphs with arbitrary number of external legs. These divergences may be absorbed by an appropriate renormalization of the low-energy coupling constants in the Lagrangian $\bar{\mathcal{L}}_4^{(Q)}$ of Eq. (24):

$$
\begin{aligned}
L_i &=& L_i^r(\mu)+\Gamma_i\lambda, \\
H_i &=& H_i^r(\mu)+\Delta_i\lambda, \\
K_i &=& K_i^r(\mu)+\Sigma_i\lambda,
\end{aligned}
\tag{34}
$$

where λ is defined as

$$
\lambda=\frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4}-\frac{1}{2}[\ln(4\pi)+\Gamma'(1)+1]\right\}
$$

with d denoting the number of space-time dimensions. The renormalized constants $L_i^r(\mu)$, $H_i^r(\mu)$, $K_i^r(\mu)$ are finite and depend on the scale μ introduced by dimensional regularization. The coefficients Γ_i, Δ_i, Σ_i are some numbers, which has to be chosen in such a way, that the generating functional (33) is finite. The resulting $Z[v, a, s, p]$ generates the general solution of the Ward identities at next-to-leading order.

We see that in order to determine the coefficients Γ_i, Δ_i, Σ_i we need to regularize the determinant of the operator D. Thus, we have to separate out the divergent part of the one-loop generating functional

$$Z_{\text{one loop}} = \frac{i}{2} \ln \det D.$$

There exists the so-called heat kernel method [33], which allows to calculate the divergent part of the $\ln \det D$. However, this method can be applied (at least, without modifications) only to the differential operators of so-called minimal kind. The operator D is nonminimal in general. It becomes minimal when the gauge parameter is set to 1: $a = 1$ (Feynman gauge). This is the case considered in Ref. [27]. Using the heat kernel method for the operator D, one obtains the divergent part of the one-loop functional [5, 27]:

$$Z^{a=1}_{\text{one loop}} = -\frac{1}{16\pi^2}\frac{1}{d-4}\int d^4x \; \mathrm{Sp}\left(\frac{1}{12}Y_{\mu\nu}Y^{\mu\nu} + \frac{1}{2}\Lambda^2\right) + \text{finite parts}, \tag{35}$$

where Sp means the trace in the flavour space η^A and $Y_{\mu\nu}$ denotes the field strength tensor of Y_μ,

$$Y_{\mu\nu} = \partial_\mu Y_\nu - \partial_\nu Y_\mu + [Y_\mu, Y_\nu].$$

One then can find the coefficients Γ_i, Δ_i, Σ_i. The coefficients Γ_i, Δ_i are listed in Ref. [6], and Σ_i in Ref. [27].

3.3 β-functions in arbitrary gauge

We are now in a position to state the main aim of the present thesis. As we have already mentioned, the coefficients Σ_i, or alternatively the β-functions, defined from Eq. (34) as

$$\beta_i = \mu \frac{dK_i^r(\mu)}{d\mu} = -\frac{1}{16\pi^2}\Sigma_i,$$

were calculated in Feynman gauge $a = 1$. We extend the evaluation of Σ_i to the case of the arbitrary gauge and formulate the problem:

Calculate the coefficients Σ_i for the arbitrary gauge parameter a.

3.3.1 Description of the method

In order to solve this problem, we chose another method of calculation of the divergent part of the one-loop functional $Z_{\text{one loop}}$ [5, 28]. We expand the determinant of D of Eq. (27) in powers of the operator δ :

$$\begin{aligned} Z_{\text{one loop}} &= \frac{i}{2}\ln\det(D_0+\delta) = \frac{i}{2}\ln\det D_0 + \frac{i}{2}\mathrm{Tr}(D_0^{-1}\delta) \\ &-\frac{i}{4}\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta) + \frac{i}{6}\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) \\ &-\frac{i}{8}\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) + \text{finite parts}, \end{aligned} \tag{36}$$

where trace Tr denotes, in coordinate space, the integral $\mathrm{Tr}\{\ldots\} = \int \mathrm{dx}\langle \mathrm{x}|\mathrm{Sp}\{\ldots\}|\mathrm{x}\rangle$. We have written out only terms, which contain the ultraviolet divergences. In momentum space at large momenta the matrix element of the operator D_0 is proportional to $1/k^2$, while the matrix element of the operator δ is proportional to k. Each trace in the sum at large momenta is proportional to the integral $\int d^4k\frac{1}{k^n}$, which is divergent only for $n \le 4$. Therefore, divergent are only traces presented in Eq. (36). We checked that in case of the minimal operator D, the expansion (36) leads to the same divergent part of Eq. (35), obtained by the heat kernel method (this was done in the strong sector, without virtual photons). Below we will use dimensional regularization as a convenient one.

To perform the calculations in the arbitrary gauge we at first explicitly expand the traces in the flavor space η^A. For the first trace we have:

$$\mathrm{Sp}\{D_0^{-1}\delta\} = (D_0^{-1}\delta)_A^A = (D_0^{-1})_{ab}(\delta)^{ba} + (D_0^{-1})_{\sigma\rho}(\delta)^{\rho\sigma},$$

where we used the fact that $(D_0^{-1})_{a\rho} = (D_0^{-1})_{\sigma b} = 0$. Inserting necessary number of completeness relation in coordinate space $\int dx|x\rangle\langle x| = 1$, we obtain

$$\mathrm{Tr}(D_0^{-1}\delta) = \int dxdy\, \left\{\langle x|(D_0^{-1})_{ab}|y\rangle\langle y|(\delta)^{ba}|x\rangle + \langle x|(D_0^{-1})_{\sigma\rho}|y\rangle\langle y|(\delta)^{\rho\sigma}|x\rangle\right\}.$$

The matrix elements of the operators D_0^{-1} and δ have the following form:

$$\begin{aligned}
\langle x|(D_0^{-1})^{ab}|y\rangle &= \delta^{ab}\Delta(x-y), \\
\langle x|(D_0^{-1})^{\mu\nu}|y\rangle &= -g^{\mu\nu}\Delta(x-y) + \Delta^{\mu\nu}(x-y), \\
\langle y|(\delta)^{AB}|x\rangle &= 2Y_\mu^{AB}(y)\partial_y^\mu\delta(x-y) + c^{AB}(y)\delta(x-y),
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
\Delta(x-y) &= \int \frac{d^dk}{(2\pi)^d}\frac{e^{-ik(x-y)}}{-k^2}, \\
\Delta^{\mu\nu}(x-y) &= (a-1)\int \frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{k^4}e^{-ik(x-y)}, \\
c(x) &= (\partial_\mu Y^\mu) + Y_\mu Y^\mu + \Lambda.
\end{aligned} \tag{38}$$

Using Eq. (37), we separate the divergent terms, that depend on the gauge parameter (through $\Delta_{\mu\nu}(x)$), and hereby obtain for $\mathrm{Tr}(D_0^{-1}\delta)$:

$$\mathrm{Tr}(D_0^{-1}\delta) = \mathrm{Tr}(D_0^{-1}\delta)^{a=1} + \int dxdy\Delta_{\sigma\rho}(x-y)\langle y|(\delta)^{\rho\sigma}|x\rangle,$$

where $\mathrm{Tr}(D_0^{-1}\delta)^{a=1}$ means the trace, which is calculated in Feynman gauge. We note, that since $Y_\mu^{\sigma\rho}(x) = 0$, then the matrix element of the operator δ simplifies to

$$\langle y|(\delta)^{\sigma\rho}|x\rangle = c^{\sigma\rho}(y)\delta(x-y). \tag{39}$$

This fact considerably reduces the number of divergent integrals, that one has to evaluate. Substituting the expression (39), partially integrating over the coordinate y and then taking integral over x, we get

$$\mathrm{Tr}(D_0^{-1}\delta) = \mathrm{Tr}(D_0^{-1}\delta)^{a=1} + \int dy\, \Delta_{\sigma\rho}(0)c^{\rho\sigma}(y).$$

The quantity $\Delta_{\sigma\rho}(0)$, which is the integral in momentum space, is zero in dimensional regularization,

$$\int \frac{d^dk}{(2\pi)^d(k^2)^m} = 0, \quad \text{for any } m; \quad \Delta_{\sigma\rho}(0) = (a-1)\int \frac{d^dk}{(2\pi)^d}\frac{k_\sigma k_\rho}{k^4} \sim g_{\sigma\rho}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2} = 0.$$

Thus, $\mathrm{Tr}(D_0^{-1}\delta) = \mathrm{Tr}(D_0^{-1}\delta)^{a=1}$. We perform the same steps for other traces in the expansion (36). Details are provided in appendix B. The divergent part of the one-loop functional in the arbitrary gauge is given by the expression

$$div Z_{\text{one loop}} = div Z^{a=1}_{\text{one loop}} + \frac{1}{4}F_0^2(1-a)\frac{1}{16\pi^2\epsilon}\int dx\Big\{\langle[H_L,\Delta_\mu]^2\rangle - \langle[H_R,\Delta_\mu]^2\rangle$$
$$+\langle H_L^2\sigma\rangle - 2\langle[H_R,\Delta_\mu]G^\mu\rangle - \frac{3}{4}\langle G_\mu G^\mu\rangle - \frac{1}{8}F_0^2 Z\langle[H_R+H_L,H_R-H_L]^2\rangle\Big\}, \tag{40}$$

where div means divergent part, $Z = C/F_0^4$ and G^μ is defined as

$$G^\mu = u^+ c_R^\mu Q u - u c_L^\mu Q u^+.$$

3.3.2 Three flavour case

Next, with help of Eq. (32), we simplify the result:

$$\langle[H_L,\Delta_\mu]^2\rangle - \langle[H_R,\Delta_\mu]^2\rangle = -\langle d^\mu\bar U^+ d_\mu\bar U Q\bar U^+ Q\bar U + d^\mu\bar U d_\mu\bar U^+ Q\bar U Q\bar U^+\rangle + 2\langle Q d_\mu\bar U Q d^\mu\bar U^+\rangle,$$
$$\langle H_L^2\sigma\rangle = \frac{1}{2}\langle(\chi\bar U^+ + \bar U\chi^+ + \chi^+\bar U + \bar U^+\chi)Q^2\rangle$$
$$-\frac{1}{2}\langle(\chi\bar U^+ + \bar U\chi^+)Q\bar U Q\bar U^+ + (\chi^+\bar U + \bar U^+\chi)Q\bar U^+ Q\bar U\rangle,$$
$$\langle[H_R,\Delta_\mu]G^\mu\rangle = -\frac{1}{2}\langle d_\mu\bar U^+[c_R^\mu Q, Q]\bar U + d_\mu\bar U[c_L^\mu Q, Q]\bar U^+\rangle$$
$$-\frac{1}{2}\langle d_\mu\bar U^+ c_R^\mu Q\bar U Q + d_\mu\bar U Q\bar U^+ c_R^\mu Q + d_\mu\bar U c_L^\mu Q\bar U^+ Q + d_\mu\bar U^+ Q\bar U c_L^\mu Q\rangle,$$
$$\langle G_\mu G^\mu\rangle = \langle c_R^\mu Q c_{R\mu} Q + c_L^\mu Q c_{L\mu} Q\rangle - 2\langle c_R^\mu Q\bar U c_{L\mu} Q\bar U^+\rangle,$$
$$\langle[H_R+H_L,H_R-H_L]^2\rangle = 32\langle Q\bar U Q\bar U^+ Q\bar U Q\bar U^+ - Q^2\bar U Q^2\bar U^+\rangle. \tag{41}$$

The second trace in the expression for $\langle[H_R,\Delta_\mu]G^\mu\rangle$ can be transformed, using partial integration and the equation of motion, obeyed by $\bar U$ [34]. We obtain

$$\langle d_\mu\bar U^+ c_R^\mu Q\bar U Q + d_\mu\bar U Q\bar U^+ c_R^\mu Q + d_\mu\bar U c_L^\mu Q\bar U^+ Q + d_\mu\bar U^+ Q\bar U c_L^\mu Q\rangle =$$
$$\langle d^\mu\bar U^+ d_\mu\bar U Q\bar U^+ Q\bar U + d^\mu\bar U d_\mu\bar U^+ Q\bar U Q\bar U^+\rangle - 2\langle Q d_\mu\bar U Q d^\mu\bar U^+\rangle$$
$$+\frac{1}{2}\langle(\chi\bar U^+ - \bar U\chi^+)Q\bar U Q\bar U^+ + (\chi^+\bar U - \bar U^+\chi)Q\bar U^+ Q\bar U\rangle$$
$$+4F_0^2 Z\langle Q\bar U Q\bar U^+ Q\bar U Q\bar U^+ - Q^2\bar U Q^2\bar U^+\rangle. \tag{42}$$

Thus, the final result for $div Z_{\text{one loop}}$ is

$$div Z_{\text{one loop}} = div Z^{a=1}_{\text{one loop}} - \frac{1}{16\pi^2(d-4)}(1-a)F_0^2\int dx\Big\{$$
$$\frac{1}{4}\langle(\chi\bar U^+ + \bar U\chi^+ + \chi^+\bar U + \bar U^+\chi)Q^2\rangle$$
$$-\frac{1}{4}\langle(\chi\bar U^+ + \bar U\chi^+)Q\bar U Q\bar U^+ + (\chi^+\bar U + \bar U^+\chi)Q\bar U^+ Q\bar U\rangle$$

$$+\frac{1}{4}\langle(\chi\bar{U}^+ - \bar{U}\chi^+)Q\bar{U}Q\bar{U}^+ + (\chi^+\bar{U} - \bar{U}^+\chi)Q\bar{U}^+Q\bar{U}\rangle$$
$$+\frac{1}{2}\langle d_\mu\bar{U}^+[c_R^\mu Q, Q]\bar{U} + d_\mu\bar{U}[c_L^\mu Q, Q]\bar{U}^+\rangle + \frac{3}{4}\langle c_R^\mu Q\bar{U}c_{L\mu}Q\bar{U}^+\rangle$$
$$-\frac{3}{8}\langle c_R^\mu Q c_{R\mu}Q + c_L^\mu Q c_{L\mu}Q\rangle\Big\}. \quad (43)$$

If we write the coefficients Σ_i as

$$\Sigma_i = \Sigma_i^{a=1} + \Sigma_i^a,$$

where $\Sigma_i^{a=1}$ are ones calculated in Feynman gauge, then Σ_i^a can be directly read off from Eq. (43). They are presented in Table 4.

i	Σ_i^a	Σ_i
1	0	$\frac{3}{4}$
2	0	Z
3	0	$-\frac{3}{4}$
4	0	$2Z$
5	0	$-\frac{9}{4}$
6	0	$\frac{3}{2}Z$
7	0	0
8	0	Z
9	$\frac{1}{4}(1-a)$	$-\frac{1}{4}a$
10	$-\frac{1}{4}(1-a)$	$\frac{3}{2}Z + \frac{1}{4}a$
11	$\frac{1}{4}(1-a)$	$\frac{1}{4}(\frac{3}{2} - a)$
12	$\frac{1}{2}(1-a)$	$\frac{1}{2}(\frac{3}{2} - a)$
13	$\frac{3}{4}(1-a)$	$\frac{3}{4}(1-a)$
14	$-\frac{3}{8}(1-a)$	$-\frac{3}{8}(1-a)$
15	0	$\frac{3}{2} + 3Z + 14Z^2$
16	0	$-3 - \frac{3}{2}Z - Z^2$
17	0	$\frac{3}{2} - \frac{3}{2}Z + 5Z^2$

Table 4: *The coefficients Σ_i and their gauge dependent parts Σ_i^a.*

3.4 Discussion of the result

Next, we would like to discuss the validity of the result. First, we checked, whether the parts Σ_i^a introduce the dependence of the physical quantities on the renormalization scale μ. From physical point of view, they should be independent of μ. We considered the masses of pions, kaons and eta meson, calculated at one-loop level with virtual photons included [27]. The expressions for them contain the following combinations of the renormalized constants $K_i^r(\mu)$:

$$\begin{aligned}
C_1 &= 6K_1^r + 6K_2^r + 5K_5^r + 5K_6^r - 6K_7^r - 15K_8^r - 5K_9^r - 23K_{10}^r - 18K_{11}^r, \\
C_2 &= K_8^r, \\
C_3 &= 12K_1^r + 12K_2^r - 18K_3^r + 9K_4^r + 10K_5^r + 10K_6^r - 12K_7^r - 12K_8^r - 10K_9^r - 10K_{10}^r, \\
C_4 &= 3K_8^r + K_9^r + K_{10}^r, \\
C_5 &= 6K_1^r + 6K_2^r + 5K_5^r + 5K_6^r - 6K_7^r - 24K_8^r - 2K_9^r - 20K_{10}^r - 18K_{11}^r, \\
C_6 &= 3K_1^r + 3K_2^r + K_5^r + K_6^r - 3K_7^r - 3K_8^r - K_9^r - K_{10}^r, \\
C_7 &= K_9^r + K_{10}^r, \\
C_8 &= 12K_1^r + 12K_2^r - 6K_3^r + 3K_4^r + 6K_5^r + 6K_6^r - 12K_7^r - 12K_8^r - 4K_9^r - 4K_{10}^r. \qquad (44)
\end{aligned}$$

Acting by the operator $\mu\frac{d}{d\mu}$ on both sides, we get the β-functions, or equivalently the Σ_i coefficients, on the right-hand sides. Using Table 4, we see that the quantities $\mu\frac{d}{d\mu}C_i$ still remain equal to zero, as they should. We also considered the amplitude of process $\pi^- K^+ \to \pi^0 K^0$ [35]. There are combinations of first six K_i^r in its expression. According to Table 4, $\mu\frac{d}{d\mu}C_i$ are zero for them. Another combinations are

$$\begin{aligned}
C_9 &= 9(M_\pi^2 + 2M_K^2)K_8^r - M_\pi^2 K_9^r + (17M_\pi^2 + 18M_K^2)K_{10}^r + 18(M_\pi^2 + M_K^2)K_{11}^r, \\
C_{10} &= K_5^r + K_6^r + 12K_8^r - 6K_{10}^r - 6K_{11}^r, \\
C_{11} &= 18K_3^r - 9K_4^r - 12K_8^r + 2K_9^r - 34K_{10}^r - 36K_{11}^r. \qquad (45)
\end{aligned}$$

They are scale independent as well. We cannot check the gauge invariance of physical quantities, since the expressions for renormalized constants $K_i^r(\mu)$ may contain parts, that do not depend on μ, but can in general depend on the gauge parameter [36].

In addition, we considered the relations between three- and two-flavour low-energy constants [37] and applied our result to them. It turns out that the scale-dependent part of the matching condition between SU(2) and SU(3) LECs is gauge-independent, as it should. Details are provided by D. Agadjanov in his thesis, in which two-flavour case is considered.

We would like to mention other approach, which was used to study the gauge dependence of constants $K_i^r(\mu)$ with $i = 1, \ldots, 14$ [36, 38]. Our result is in agreement with the findings of Refs. [36, 38]. We also mention Ref. [39], in which the author provides the expression for the divergent part of the one-loop functional for operators of non-minimal kind. The expression given in that paper is not suited for a direct application to ChPT.

4 Conclusion

In present thesis we calculated the one-loop generating functional for mesons and virtual photons in case of arbitrary gauge. The problem was that the usual heat kernel method cannot be applied. Therefore, we chose another method, which allowed to solve the problem. Then we evaluated the β-functions of the electromagnetic low-energy constants. After that, we considered different checks on our result. To that end, we checked, if the β-functions introduce the dependence of the physical quantities on renormalization scale. We took the masses of pions, kaons, eta meson and amplitude of $\pi^- K^+ \to \pi^0 K^0$ process. In all these cases we found scale independence, as it should. We also checked the relations between three- and two-flavour β-functions. They were all valid. Further, we compared our β-functions with ones, obtained by another method and concluded that they coincide.

In future, we plan to study the gauge dependence of the scale-independent parts of the constants $K_i^r(\mu)$. We also plan to consider the problem in context of lattice QCD calculation of the low-energy constants.

Acknowledgments

I am grateful to J. Gasser for suggesting the problem, which became the topic of this thesis. I thank A. Rusetsky for his excellent supervision. Further, I thank B. Ananthanarayan for useful discussions and A. Khelashvili for constant help and support during the work.

References

[1] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973).

[2] S. Weinberg, Phys. Rev. Lett. **31**, 494 (1973).

[3] H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. B **47**, 365 (1973).

[4] S. Scherer, M.R. Schindler, arXiv:hep-ph/0505265v1.

[5] J. Gasser and H. Leutwyler, Ann. Phys. **158** (1984) 142.

[6] J. Gasser and H. Leutwyler, Nucl. Phys. **B250** (1985) 465.

[7] H. Leutwyler, Annals Phys. **235**, 165 (1994).

[8] S. Weinberg, Physica A **96**, 327 (1979).

[9] C. Vafa and E. Witten, Nucl. Phys. **B234**, 173 (1984).

[10] S. Coleman, J. Math. Phys. **7**, 787 (1966).

[11] J. Goldstone, Nuovo Cim. **19**, 154 (1961).

[12] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[13] A.V. Manohar, *Effective Field Theories, Lectures at the 1996 Schladming Winter School* [arxiV:hep-ph/9606222v1].

[14] S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969).

[15] C. G. Callan, S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1969).

[16] G. Ecker, Prog. Part. Nucl. Phys. **35**, 1 (1995) [arXiv:hep-ph/9501357].

[17] T. Ebertshäuser, *Mesonic Chiral Perturbation Theory: Odd Intrinsic Parity Sector*, PhD thesis, Johannes GutenbergUniversität, Mainz, Germany, 2001, http://archimed.unimainz.de/.

[18] H. Georgi, *Weak Interactions and Modern Particle Theory* (Benjamin/ Cummings, Menlo Park, 1984).

[19] S. Pislak et al. [BNL-E865 Collaboration], Phys. Rev. Lett. **87**, 221801 (2001).

[20] G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Rev. Lett. 86, 5008 (2001).

[21] A.V. Manohar and H. Georgi, Nucl. Phys. **B234**, (1984) 189.

[22] M. Soldate and R. Sundrum, Nucl. Phys. **B340**, (1990) 1.

[23] R.S. Chivukula, M.J. Dugan, and M. Golden. Phys. Rev. **D47**, (1993) 2930.

[24] S. Weinberg, Phys. Rev. **118**, 838 (1960).

[25] A. Pich, Rept. Prog. Phys. **58**, 563 (1995).

[26] G. Ecker, J. Gasser, A. Pich, E.de Rafael, Nucl. Phys. **B321** (1989) 311.

[27] R. Urech, Nucl. Phys. **B433** (1995) 234 [arXiv:hep-ph/9405341].

[28] J. Schweizer, JHEP **0302**, 007 (2003) [arXiv:hep-ph/0212188].

[29] R.P. Feynman Acta Phys. Polon. **24** (1963) 697.

[30] B.S. DeWitt. Phys. Rev. **162**, 1195 (1967).

[31] M. Knecht and R. Urech, Nucl. Phys. **B519** (1998) 329 [arXiv:hep-ph/9709348].

[32] P. Ramond, *Field Theory: A Modern Primer*, Benjamin, London/New York, 1981.

[33] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[34] H. Neufeld and H. Rupertsberger, Z. Phys. C **71**, 131 (1996).

[35] A. Nehme and P. Talavera, Phys. Rev. D **65** (2002) 054023 [arXiv:hep-ph/0107299].

[36] B. Moussallam, Nucl. Phys. B **504**, 381 (1997) [arXiv:hep-ph/9701400v2].

[37] C. Haefeli, M. A. Ivanov and M. Schmid, Eur. Phys. J. C **53** 549 (2008) [arXiv:0710.5432 [hep-ph]].

[38] B. Ananthanarayan and B. Moussallam, JHEP **0406** (2004) 047 [arXiv:hep-ph/0405206].

[39] B.Ananthanarayan, J. Phys. **A41** (2008) [arXiv:0808.2781v1 [hep-th]].

Appendices

A Integrals

In this appendix we collect the divergent parts of necessary integrals, that are needed during the calculations with help of dimensional regularization. We note that $d = 4 - 2\epsilon$.

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{[p^2 - 2kp + m^2]^3} = \frac{1}{2\Gamma(3)} \frac{i}{16\pi^2 \epsilon} g^{\mu\nu} + \text{f. p.},$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu p^\lambda}{[p^2 - 2kp + m^2]^3} = \frac{1}{2\Gamma(3)} \frac{i}{16\pi^2 \epsilon} [g^{\mu\nu} k^\lambda + g^{\mu\lambda} k^\nu + g^{\nu\lambda} k^\mu] + \text{f. p.},$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu p^\lambda p^\rho}{[p^2 - 2kp + m^2]^3} = \frac{1}{2\Gamma(3)} \frac{i}{16\pi^2 \epsilon} \Big\{ \frac{1}{2} [g^{\mu\nu} k^\rho k^\lambda + g^{\mu\lambda} k^\rho k^\nu + g^{\nu\lambda} k^\mu k^\rho + g^{\mu\rho} k^\nu k^\lambda +$$
$$+ g^{\nu\rho} k^\mu k^\lambda + g^{\lambda\rho} k^\mu k^\nu] - \frac{1}{4} (m^2 - k^2) [g^{\mu\nu} g^{\rho\lambda} + g^{\mu\lambda} g^{\rho\nu} + g^{\nu\lambda} g^{\mu\rho}] \Big\} + \text{f. p.},$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu p^\lambda p^\rho}{[p^2 - 2kp + m^2]^4} = \frac{1}{4\Gamma(4)} \frac{i}{16\pi^2 \epsilon} [g^{\mu\nu} g^{\rho\lambda} + g^{\mu\lambda} g^{\rho\nu} + g^{\nu\lambda} g^{\mu\rho}] + \text{f. p.},$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu p^\lambda p^\rho p^\sigma}{[p^2 - 2kp + m^2]^4} = \frac{1}{4\Gamma(4)} \frac{i}{16\pi^2 \epsilon} [(g^{\mu\nu} g^{\sigma\rho} + g^{\mu\rho} g^{\sigma\nu} + g^{\nu\rho} g^{\sigma\mu}) k^\lambda$$
$$+ (g^{\mu\nu} g^{\sigma\lambda} + g^{\mu\lambda} g^{\sigma\nu} + g^{\nu\lambda} g^{\mu\sigma}) k^\rho + (g^{\mu\lambda} g^{\sigma\rho} + g^{\mu\rho} g^{\lambda\sigma} + g^{\lambda\rho} g^{\mu\sigma}) k^\nu$$
$$+ (g^{\nu\lambda} g^{\rho\sigma} + g^{\nu\rho} g^{\sigma\lambda} + g^{\lambda\rho} g^{\sigma\nu}) k^\mu + (g^{\mu\nu} g^{\rho\lambda} + g^{\mu\lambda} g^{\rho\nu} + g^{\nu\lambda} g^{\mu\rho}) k^\sigma] + \text{f. p.},$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu p^\lambda p^\rho p^\sigma p^\epsilon}{[p^2 - 2kp + m^2]^5} = \frac{1}{8\Gamma(5)} \frac{i}{16\pi^2 \epsilon} [(g^{\mu\nu} g^{\sigma\rho} + g^{\mu\rho} g^{\sigma\nu} + g^{\nu\rho} g^{\sigma\mu}) g^{\lambda\epsilon}$$
$$+ (g^{\mu\nu} g^{\sigma\lambda} + g^{\mu\lambda} g^{\sigma\nu} + g^{\nu\lambda} g^{\mu\sigma}) g^{\rho\epsilon} + (g^{\mu\lambda} g^{\sigma\rho} + g^{\mu\rho} g^{\lambda\sigma} + g^{\lambda\rho} g^{\mu\sigma}) g^{\nu\epsilon}$$
$$+ (g^{\nu\lambda} g^{\rho\sigma} + g^{\nu\rho} g^{\sigma\lambda} + g^{\lambda\rho} g^{\sigma\nu}) g^{\mu\epsilon} + (g^{\mu\nu} g^{\rho\lambda} + g^{\mu\lambda} g^{\rho\nu} + g^{\nu\lambda} g^{\mu\rho}) g^{\sigma\epsilon}] + \text{f. p.},$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu p^\lambda p^\rho p^\sigma p^\epsilon k^\alpha k^\beta}{[p^2 - 2kp + m^2]^6} = \frac{1}{16\Gamma(6)} \frac{i}{16\pi^2 \epsilon} [$$
$$(g^{\mu\nu} g^{\sigma\rho} + g^{\mu\rho} g^{\sigma\nu} + g^{\nu\rho} g^{\sigma\mu})(g^{\alpha\beta} g^{\lambda\epsilon} + g^{\alpha\lambda} g^{\beta\epsilon} + g^{\alpha\epsilon} g^{\beta\lambda})$$
$$+ (g^{\mu\nu} g^{\sigma\lambda} + g^{\mu\lambda} g^{\sigma\nu} + g^{\nu\lambda} g^{\mu\sigma})(g^{\alpha\beta} g^{\rho\epsilon} + g^{\alpha\rho} g^{\beta\epsilon} + g^{\alpha\epsilon} g^{\beta\rho})$$
$$+ (g^{\mu\lambda} g^{\sigma\rho} + g^{\mu\rho} g^{\lambda\sigma} + g^{\lambda\rho} g^{\mu\sigma})(g^{\alpha\beta} g^{\nu\epsilon} + g^{\alpha\nu} g^{\beta\epsilon} + g^{\alpha\epsilon} g^{\beta\nu})$$
$$+ (g^{\nu\lambda} g^{\rho\sigma} + g^{\nu\rho} g^{\sigma\lambda} + g^{\lambda\rho} g^{\sigma\nu})(g^{\alpha\beta} g^{\mu\epsilon} + g^{\alpha\mu} g^{\beta\epsilon} + g^{\alpha\epsilon} g^{\beta\mu})$$
$$+ (g^{\mu\nu} g^{\rho\lambda} + g^{\mu\lambda} g^{\rho\nu} + g^{\nu\lambda} g^{\mu\rho})(g^{\alpha\beta} g^{\sigma\epsilon} + g^{\alpha\sigma} g^{\beta\epsilon} + g^{\alpha\epsilon} g^{\beta\sigma})$$

$$+(g^{\alpha\sigma}g^{\beta\rho}+g^{\beta\sigma}g^{\alpha\rho})(g^{\mu\nu}g^{\epsilon\lambda}+g^{\mu\epsilon}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\epsilon})$$
$$+(g^{\epsilon\sigma}g^{\alpha\rho}+g^{\alpha\sigma}g^{\epsilon\rho})(g^{\mu\nu}g^{\beta\lambda}+g^{\mu\beta}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\beta})$$
$$+(g^{\epsilon\sigma}g^{\beta\rho}+g^{\beta\sigma}g^{\epsilon\rho})(g^{\mu\nu}g^{\alpha\lambda}+g^{\mu\alpha}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\alpha})$$
$$+(g^{\alpha\lambda}g^{\beta\nu}+g^{\beta\lambda}g^{\alpha\nu})(g^{\mu\rho}g^{\epsilon\sigma}+g^{\mu\epsilon}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\epsilon})$$
$$+(g^{\epsilon\lambda}g^{\alpha\nu}+g^{\alpha\lambda}g^{\epsilon\nu})(g^{\mu\rho}g^{\beta\sigma}+g^{\mu\beta}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\beta})$$
$$+(g^{\epsilon\lambda}g^{\beta\nu}+g^{\beta\lambda}g^{\epsilon\nu})(g^{\mu\rho}g^{\alpha\sigma}+g^{\mu\alpha}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\alpha})$$
$$+(g^{\alpha\lambda}g^{\beta\mu}+g^{\beta\lambda}g^{\alpha\mu})(g^{\nu\rho}g^{\epsilon\sigma}+g^{\nu\sigma}g^{\rho\epsilon})+(g^{\epsilon\lambda}g^{\alpha\mu}+g^{\alpha\lambda}g^{\epsilon\mu})(g^{\nu\rho}g^{\beta\sigma}+g^{\nu\sigma}g^{\rho\beta})$$
$$+(g^{\epsilon\lambda}g^{\beta\mu}+g^{\beta\lambda}g^{\epsilon\mu})(g^{\nu\rho}g^{\alpha\sigma}+g^{\nu\sigma}g^{\rho\alpha})+(g^{\alpha\mu}g^{\beta\nu}+g^{\beta\mu}g^{\alpha\nu})(g^{\lambda\rho}g^{\epsilon\sigma}+g^{\lambda\sigma}g^{\rho\epsilon})$$
$$+(g^{\epsilon\mu}g^{\alpha\nu}+g^{\alpha\mu}g^{\epsilon\nu})(g^{\lambda\rho}g^{\beta\sigma}+g^{\lambda\sigma}g^{\rho\beta})+(g^{\epsilon\mu}g^{\beta\nu}+g^{\beta\mu}g^{\epsilon\nu})(g^{\lambda\rho}g^{\alpha\sigma}+g^{\lambda\sigma}g^{\rho\alpha})]+\mathrm{f.\,p.}$$

B Calculation of traces

Below, we present the details of calculations.

B.1 $\mathrm{Tr}(\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta)$

We expand the second trace:

$$\begin{aligned}\mathrm{Sp}\{D_0^{-1}\delta D_0^{-1}\delta\} &= (D_0^{-1})_{ab}(\delta)^{bc}(D_0^{-1})_{cd}(\delta)^{da}+2(D_0^{-1})_{ab}(\delta)^{b\sigma}(D_0^{-1})_{\sigma\rho}(\delta)^{\rho a}\\ &\quad +(D_0^{-1})_{\sigma\rho}(\delta)^{\rho\mu}(D_0^{-1})_{\mu\nu}(\delta)^{\nu\sigma}\end{aligned}\tag{B.1}$$

Analogously, we separate the divergent terms, that depend on the gauge parameter (through $\Delta_{\mu\nu}(x)$), and hereby obtain for $\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta)$:

$$\begin{aligned}\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta) &= \mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta)^{a=1}\\ &\quad +2\int dxdydzdu\,\Delta(x-y)\Delta_{\sigma\rho}(z-u)\langle y|(\delta)^{a\sigma}|z\rangle\langle u|(\delta)^{\rho a}|x\rangle\\ &\quad -2g_{\sigma\rho}\int dxdydzdu\,\Delta(x-y)\Delta_{\mu\nu}(z-u)\langle y|(\delta)^{\rho\mu}|z\rangle\langle u|(\delta)^{\nu\sigma}|x\rangle\\ &\quad +\int dxdydzdu\,\Delta_{\sigma\rho}(x-y)\Delta_{\mu\nu}(z-u)\langle y|(\delta)^{\rho\mu}|z\rangle\langle u|(\delta)^{\nu\sigma}|x\rangle\\ &= \mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta)^{a=1}+2I+2K+L,\end{aligned}\tag{B.2}$$

Thus, it is necessary to find the divergent parts of the integrals I, K, L. The integral I is

$$\begin{aligned}I &= \int dxdydzdu\,\Delta(x-y)\Delta_{\sigma\rho}(z-u)[2Y_\mu(y)\partial_y^\mu\delta(y-z)+c(y)\delta(y-z)]^{a\sigma}\times\\ &\quad\times[2Y_\nu(u)\partial_u^\nu\delta(u-x)+c(u)\delta(u-x)]^{\rho a}\\ &= \int dydu\,[-2Y_\mu(y)\partial_y^\mu\Delta(u-y)+b(y)\Delta(u-y)]^{a\sigma}\times\\ &\quad\times[-2Y_\nu(u)\partial_u^\nu\Delta_{\sigma\rho}(y-u)+b(u)\Delta_{\sigma\rho}(y-u)]^{\rho a}=I_1+I_2+I_3+I_4,\end{aligned}\tag{B.3}$$

where the following matrix is introduced:

$$b(x) = Y_\mu Y^\mu + \Lambda - (\partial_\mu Y^\mu). \tag{B.4}$$

as well as notations for the integrals:

$$\begin{aligned}
I_1 &= 4\int dydu\, Y^{a\sigma}_\mu(y) Y^{\rho a}_\nu(u) I^\mu_{1\,\sigma\rho}, \\
I_2 &= -2\int dydu\, Y^{a\sigma}_\mu(y) b^{\rho a}(u) I^{\mu\nu}_{2\,\sigma\rho}, \\
I_3 &= -2\int dydu\, Y^{a\sigma}_\mu(y) b^{\rho a}(u) I^{\mu}_{3\,\sigma\rho}, \\
I_4 &= \int dydu\, b^{a\sigma}(y) b^{\rho a}(u) I_{4\,\sigma\rho},
\end{aligned} \tag{B.5}$$

with

$$\begin{aligned}
I_1^{\mu\nu\sigma\rho} &= \partial^\mu_y \Delta(u-y)\partial^\nu_u \Delta^{\sigma\rho}(y-u), \\
I_2^{\mu\sigma\rho} &= \partial^\mu_y \Delta(u-y)\Delta^{\sigma\rho}(y-u), \\
I_3^{\mu\sigma\rho} &= \Delta(u-y)\partial^\mu_u \Delta^{\sigma\rho}(y-u), \\
I_4^{\sigma\rho} &= \Delta(u-y)\Delta^{\sigma\rho}(y-u).
\end{aligned} \tag{B.6}$$

The Lorentz indices are raised and lowered by the metric tensor $g^{\mu\nu}$. With the help of Eq.(38) we write the integrals (B.6) in momentum space. For the first integral we have

$$I_1^{\mu\nu\sigma\rho} = (a-1)\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{k_1^\nu k_1^\sigma k_1^\rho k_2^\mu}{k_1^4 k_2^2} e^{-i(k_2-k_1)(u-y)}.$$

Introducing new integration variables

$$(k_1, k_2) \mapsto (k_1, p): \quad p = k_2 - k_1,$$

with the transformation Jacobian $J = 1$ and Feynman parametrization, the integral $I_1^{\mu\nu\sigma\rho}$ takes the form

$$I_1^{\mu\nu\sigma\rho} = (a-1)\int \frac{d^dp}{(2\pi)^d} e^{-ip(u-y)} F_1^{\mu\nu\sigma\rho},$$

where

$$F_1^{\mu\nu\sigma\rho} = 2!\int_0^1 dx \int_0^x dy \int \frac{d^dk_1}{(2\pi)^d}\frac{k_1^\nu k_1^\sigma k_1^\rho p^\mu + k_1^\nu k_1^\sigma k_1^\rho k_1^\mu}{[k_1^2 + 2(1-x)(k_1 p) + p^2(1-x)]^3}.$$

Here and below we use the values of integrals, presented in appendix A; we obtain $F_1^{\mu\nu\sigma\rho}$:

$$\begin{aligned}
F_1^{\mu\nu\sigma\rho} = \frac{1}{12}\frac{i}{16\pi^2\epsilon}\Big\{\frac{1}{2}[-g^{\nu\sigma}p^\mu p^\rho - g^{\nu\rho}p^\mu p^\sigma - g^{\sigma\rho}p^\mu p^\nu + g^{\mu\nu}p^\sigma p^\rho + g^{\mu\sigma}p^\nu p^\rho + g^{\mu\rho}p^\nu p^\sigma] \\
-\frac{1}{4}[g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\sigma\nu} + g^{\nu\rho}g^{\sigma\mu}]\Big\} + \text{f. p.}
\end{aligned} \tag{B.7}$$

Then, integrating over the momenta p with help of the formulas

$$\int \frac{d^d p}{(2\pi)^d} p^\mu p^\nu e^{-ip(u-y)} = -\partial_u^\mu \partial_u^\nu \delta(u-y), \quad \int \frac{d^d p}{(2\pi)^d} p^2 e^{-ip(u-y)} = -\partial_u^2 \delta(u-y),$$

we get $I_1^{\mu\nu\sigma\rho}$ and consequently I_1, which reads

$$I_1 = -2(a-1)\frac{i}{16\pi^2\epsilon}\int dx X^a \partial^2 X^a + \text{f. p.},$$

where the result for I_1 was simplified by summation over Lorentz indices and the use of property of the matrix Y_μ: $Y_\mu^{a\sigma} = -Y_\mu^{\sigma a} = X^a \delta_\mu^\sigma$; the quantity X^a is defined in Eq. (31). In the same manner we calculate the divergent parts of the other integrals in Eq. (B.6). We obtain

$$\begin{aligned}
I_2 &= -\frac{1}{2}(a-1)\frac{i}{16\pi^2\epsilon}\int dx X^a \partial_\rho b^{\rho a} + \text{f. p.}, \\
I_3 &= -(a-1)\frac{i}{16\pi^2\epsilon}\int dx X^a \partial_\sigma b^{a\sigma} + \text{f. p.}, \\
I_4 &= -\frac{1}{4}(a-1)\frac{i}{16\pi^2\epsilon}\int dx g_{\sigma\rho} b^{a\sigma} b^{\rho a} + \text{f. p.}
\end{aligned} \tag{B.8}$$

The values of integrals from appendix A as well as the following additional formula were used:

$$\int \frac{d^d p}{(2\pi)^d} p^\mu e^{-ip(u-y)} = i\partial_u^\mu \delta(u-y).$$

The final expression for the integral $I = I_1 + I_2 + I_3 + I_4$ can be reduced to a more simple one, if we replace $b^{a\sigma}$ by $b^{\sigma a}$. As it follows from the definition of the matrix $b(x)$,

$$b^{a\sigma} = b^{\sigma a} - 2\partial^\mu Y_\mu^{a\sigma} = b^{\sigma a} - 2\partial^\sigma X^a.$$

Therefore, we get

$$I = -(a-1)\frac{i}{16\pi^2\epsilon}\int dx \left\{-2b^{\rho a}\partial_\rho X^a + \frac{1}{4} g_{\sigma\rho} b^{\sigma a} b^{\rho a}\right\} + \text{f. p.}$$

The second integral K with help of the Eq. (39), takes the form

$$\begin{aligned}
K &= -g_{\sigma\rho}\int dx dy dz du\, \Delta(x-y)\Delta_{\mu\nu}(z-u) c^{\rho\mu}(y) c^{\nu\sigma}(u)\delta(y-z)\delta(u-x) \\
&= -g_{\sigma\rho}\int dy du\, \Delta(u-y)\Delta_{\mu\nu}(y-u) c^{\rho\mu}(y) c^{\nu\sigma}(u).
\end{aligned} \tag{B.9}$$

To calculate its divergent part we write

$$K^{\mu\nu} = I_4^{\mu\nu} = -2!(a-1)\int \frac{d^d p}{(2\pi)^d} e^{-ip(u-y)} \int_0^1 dx \int_0^x dy \int \frac{d^d k_1}{(2\pi)^d} \frac{k_1^\mu k_1^\nu}{[k_1^2 + 2(1-x)(k_1 p) + p^2(1-x)]^3},$$

$$K^{\mu\nu} = -\frac{1}{4}(a-1)\frac{i}{16\pi^2\epsilon}g^{\mu\nu}\delta(u-y) + \mathrm{f.\,p.}$$

Then,

$$K = \frac{1}{4}(a-1)\frac{i}{16\pi^2\epsilon}\int dx g_{\mu\nu}g_{\sigma\rho}c^{\rho\mu}(x)c^{\nu\sigma}(x) + \mathrm{f.\,p.}$$

The matrix element $c^{\rho\sigma}$ is

$$c^{\rho\sigma} = -(\rho + X^a X^a)g^{\rho\sigma},$$

and thus we obtain

$$K = (a-1)\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^a X^a)^2 + \mathrm{f.\,p.}$$

The last integral L can be written as

$$L = \int dydu\, c^{\rho\mu}(y)c^{\nu\sigma}(u)L_{\sigma\rho\mu\nu},$$

where

$$L^{\sigma\rho\mu\nu} = \Delta^{\sigma\rho}(u-y)\Delta^{\mu\nu}(y-u)$$

We have

$$L^{\sigma\rho\mu\nu} = (a-1)^2\int\frac{d^dp}{(2\pi)^d}e^{-ip(u-y)}F^{\sigma\rho\mu\nu},$$

with

$$\begin{aligned} F^{\sigma\rho\mu\nu} &= \int\frac{d^dk_1}{(2\pi)^d}\frac{k_1^\mu k_1^\nu(k_1^\sigma + p^\sigma)(k_1^\rho + p^\rho)}{k_1^4(k_1+p)^4} \\ &= 3!\int_0^1 dx\int_0^x dy\int_0^y dz\int\frac{d^dk_1}{(2\pi)^d}\frac{k_1^\mu k_1^\nu k_1^\sigma k_1^\rho}{[k_1^2 + 2(1-y)(k_1p) + p^2(1-y)]^4} + \mathrm{f.\,p.} \end{aligned} \tag{B.10}$$

Taking the divergent part of the integral in momentum space we get $L^{\sigma\rho\mu\nu}$:

$$L^{\sigma\rho\mu\nu} = \frac{1}{24}(a-1)^2\frac{i}{16\pi^2\epsilon}[g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}]\delta(u-y) + \mathrm{f.\,p.}$$

The integral L becomes

$$L = \frac{1}{24}(a-1)^2\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^a X^a)^2 g_{\mu\nu}g_{\sigma\rho}[g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}] + \mathrm{f.\,p.}$$

$$L = (a-1)^2\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^a X^a)^2 + \mathrm{f.\,p.}$$

Thus, the divergent part of the second trace is

$$\begin{aligned} &div\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta) = div\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta)^{a=1} \\ &-2(a-1)\frac{i}{16\pi^2\epsilon}\int dx\left\{-2b^{\rho a}\partial_\rho X^a + \frac{1}{4}g_{\sigma\rho}b^{\sigma a}b^{\rho a}\right\} \\ &+\left\{2(a-1) + (a-1)^2\right\}\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^a X^a)^2, \end{aligned} \tag{B.11}$$

where div means the divergent part.

B.2 $\mathrm{Tr}(\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta)$

Next we consider the third trace $\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)$. We have:

$$\begin{aligned}
&\mathrm{Sp}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) = (D_0^{-1})_{ab}(\delta)^{bc}(D_0^{-1})_{cd}(\delta)^{de}(D_0^{-1})_{ef}(\delta)^{fa} \\
&+3(D_0^{-1})_{ab}(\delta)^{bc}(D_0^{-1})_{cd}(\delta)^{d\sigma}(D_0^{-1})_{\sigma\rho}(\delta)^{\rho a} + (D_0^{-1})_{\sigma\rho}(\delta)^{\rho\lambda}(D_0^{-1})_{\lambda\mu}(\delta)^{\mu a}(D_0^{-1})_{ab}(\delta)^{b\sigma} \\
&+(D_0^{-1})_{\sigma\rho}(\delta)^{\rho\lambda}(D_0^{-1})_{\lambda\mu}(\delta)^{\mu\nu}(D_0^{-1})_{\nu\epsilon}(\delta)^{\epsilon\sigma}
\end{aligned} \tag{B.12}$$

We omit the last term, since it produces finite integral, due to Eq. (39). Then,

$$\begin{aligned}
&\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) - \mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)^{a=1} = 3M + 3N + \mathrm{f.\,p.} \\
&= 3\int dxdydzdtdudv\,\Delta(x-y)\Delta(z-t)\Delta_{\sigma\rho}(u-v)\langle y|(\delta)^{ab}|z\rangle\langle t|(\delta)^{b\sigma}|u\rangle\langle v|(\delta)^{\rho a}|x\rangle \\
&+3\int dxdydzdtdudv\,\{-\Delta(x-y)\Delta_{\lambda\mu}(z-t)\Delta(u-v)g_{\sigma\rho} - \Delta(x-y)_{\sigma\rho}\Delta(z-t)\Delta(u-v)g_{\lambda\mu} \\
&+\Delta_{\sigma\rho}(x-y)\Delta_{\lambda\mu}(z-t)\Delta(u-v)\}\langle y|(\delta)^{\rho\lambda}|z\rangle\langle t|(\delta)^{\mu a}|u\rangle\langle v|(\delta)^{a\sigma}|x\rangle + \mathrm{f.\,p.}
\end{aligned} \tag{B.13}$$

The integral M is of the form

$$\begin{aligned}
&M = \int dydtdv\,[-2Y_\mu(y)\partial_y^\mu\Delta(v-y) + b(y)\Delta(v-y)]^{ab}\times \\
&\times[-2Y_\nu(t)\partial_t^\nu\Delta(y-t) + b(t)\Delta(y-t)]^{b\sigma}[-2Y_\lambda(v)\partial_t^\lambda\Delta_{\sigma\rho}(t-v) + b(v)\Delta_{\sigma\rho}(t-v)]^{\rho a} \\
&= M_1 + M_2 + M_3 + M_4 + \mathrm{f.\,p.},
\end{aligned} \tag{B.14}$$

where

$$\begin{aligned}
M_1 &= -8\int dydtdv\,Y_\mu^{ab}(y)Y_\nu^{b\sigma}(t)Y_\lambda^{\rho a}(v)M_{1\sigma\rho}^{\mu\nu\lambda}, \\
M_2 &= 4\int dydtdv\,Y_\mu^{ab}(y)Y_\nu^{b\sigma}(t)b^{\rho a}(v)M_{2\sigma\rho}^{\mu\nu}, \\
M_3 &= 4\int dydtdv\,Y_\mu^{ab}(y)Y_\nu^{\rho a}(v)b^{b\sigma}(t)M_{3\sigma\rho}^{\mu\nu}, \\
M_4 &= 4\int dydtdv\,Y_\mu^{b\sigma}(t)Y_\nu^{\rho a}(v)b^{ab}(y)M_{4\sigma\rho}^{\mu\nu},
\end{aligned} \tag{B.15}$$

with

$$\begin{aligned}
M_1^{\mu\nu\lambda\sigma\rho} &= \partial_y^\mu\Delta(v-y)\partial_t^\nu\Delta(y-t)\partial_v^\lambda\Delta^{\sigma\rho}(t-v), \\
M_2^{\mu\nu\sigma\rho} &= \partial_y^\mu\Delta(v-y)\partial_t^\nu\Delta(y-t)\Delta^{\sigma\rho}(t-v), \\
M_3^{\mu\nu\sigma\rho} &= \partial_y^\mu\Delta(v-y)\Delta(y-t)\partial_v^\nu\Delta^{\sigma\rho}(t-v), \\
M_4^{\mu\nu\sigma\rho} &= \Delta(v-y)\partial_t^\mu\Delta(y-t)\partial_v^\nu\Delta^{\sigma\rho}(t-v).
\end{aligned} \tag{B.16}$$

To calculate the divergent part of $M_1^{\mu\nu\lambda\sigma\rho}$, we write

$$M_1^{\mu\nu\lambda\sigma\rho} = i^3(a-1)\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\frac{k_1^\mu k_2^\nu k_3^\lambda k_3^\sigma k_3^\rho}{k_1^2k_2^2k_3^4}e^{-ik_1(v-y)-ik_2(y-t)-ik_3(t-v)} \tag{B.17}$$

Then we similarly introduce the new integration variables:

$$(k_1,k_2,k_3)\mapsto(p,q,k): \quad p=k_1-k_3,\ q=k_2-k_3,\ k=k_3,$$

with Jacobian $J=1$, and obtain

$$M_1^{\mu\nu\lambda\sigma\rho} = i^3(a-1)\int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}e^{ip(y-v)+iq(t-y)}G_1^{\mu\nu\lambda\sigma\rho},$$

where

$$G_1^{\mu\nu\lambda\sigma\rho} = 3!\int_0^1 dx\int_0^x dy\int_0^y dz\int\frac{d^dk}{(2\pi)^d}\frac{p^\mu k^\nu k^\lambda k^\sigma k^\rho + q^\nu k^\mu k^\lambda k^\sigma k^\rho + k^\mu k^\nu k^\lambda k^\sigma k^\rho + \text{f. p.}}{[k^2+2k[p(x-y)+q(1-x)]+p^2(x-y+q^2(1-z))]^4}$$

After integrations, we obtain

$$\begin{aligned}G_1^{\mu\nu\lambda\sigma\rho} &= \frac{1}{4}\frac{i}{16\pi^2\epsilon}\frac{1}{24}\{(3p^\mu-q^\mu)(g^{\nu\lambda}g^{\rho\sigma}+g^{\nu\rho}g^{\sigma\lambda}+g^{\lambda\rho}g^{\sigma\nu})\\&+(3q^\nu-p^\nu)(g^{\mu\lambda}g^{\sigma\rho}+g^{\mu\rho}g^{\lambda\sigma}+g^{\lambda\rho}g^{\mu\sigma})-(p^\lambda+q^\lambda)(g^{\mu\nu}g^{\sigma\rho}+g^{\mu\rho}g^{\sigma\nu}+g^{\nu\rho}g^{\sigma\mu})\\&-(p^\rho+q^\rho)(g^{\mu\nu}g^{\sigma\lambda}+g^{\mu\lambda}g^{\sigma\nu}+g^{\nu\lambda}g^{\mu\sigma})-(p^\sigma+q^\sigma)(g^{\mu\nu}g^{\rho\lambda}+g^{\mu\lambda}g^{\rho\nu}+g^{\nu\lambda}g^{\mu\rho})\}+\text{f. p.}\end{aligned}$$

Then, integrating over p,q according to the formulas

$$\int\frac{d^dp}{(2\pi)^d}p^\mu e^{ip(y-v)} = -i\partial_y^\mu\delta(y-v), \quad \int\frac{d^dq}{(2\pi)^d}q^\nu e^{iq(t-y)} = i\partial_y^\nu\delta(t-y),$$

we get $M_1^{\mu\nu\lambda\sigma\rho}$. After summation over Lorentz indices and performing of the necessary integrations, the divergent part of M_1 reads

$$M_1 = 2(a-1)\frac{i}{16\pi^2\epsilon}\int dx Y_\mu^{ab}X^a\partial^\mu X^b + \text{f. p.}$$

The integral $M_2^{\mu\nu\sigma\rho}$is

$$M_2^{\mu\nu\sigma\rho} = i^2(a-1)\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}e^{ip(y-v)+iq(t-y)}G_2^{\mu\nu\sigma\rho},$$

where

$$G_2^{\mu\nu\sigma\rho} = 3!\int_0^1 dx\int_0^x dy\int_0^y dz\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu k^\sigma k^\rho}{[k^2+\ldots]^4} + \text{f. p.}$$

Thus,

$$M_2^{\mu\nu\sigma\rho} = -\frac{1}{24}(a-1)\frac{i}{16\pi^2\epsilon}[g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}]\delta(y-v)\delta(t-y) + \mathrm{f.\,p.}$$

We note that the other integrals $M_3^{\mu\nu\sigma\rho}, M_4^{\mu\nu\sigma\rho}$ have the same divergent part. Therefore, we get

$$\begin{aligned} M_2 + M_3 + M_4 &= -\frac{1}{6}(a-1)\frac{i}{16\pi^2\epsilon}\int dx \left[Y_\mu^{ab}Y_\nu^{b\sigma}b^{\rho a} + Y_\mu^{ab}Y_\nu^{\rho a}b^{b\sigma} + Y_\mu^{b\sigma}Y_\nu^{\rho a}b^{ab}\right] \times \\ &\quad \times[g^{\mu\nu}g_{\sigma\rho} + \delta_\rho^\mu\delta_\sigma^\nu + \delta_\sigma^\mu\delta_\rho^\nu] + \mathrm{f.\,p.}, \end{aligned} \quad \text{(B.18)}$$

$$M_2 + M_3 + M_4 = -\frac{1}{6}(a-1)\frac{i}{16\pi^2\epsilon}\int dx \left\{6Y_\rho^{ab}X^b(b^{\rho a} + b^{a\rho}) - 24b^{ab}X^aX^b\right\} + \mathrm{f.\,p.}$$

The integral N, taking into account Eq. (39), can be written as

$$N = N_1 + N_2 + N_3 + \mathrm{f.\,p.},$$

where

$$\begin{aligned} N_1 &= -4\int dydtdv\, g_{\sigma\rho}c^{\rho\lambda}(y)Y_\nu^{\mu a}(t)Y_\epsilon^{a\sigma}(v)N_{1\lambda\mu}^{\nu\epsilon}, \\ N_2 &= -4\int dydtdv\, g_{\lambda\mu}c^{\rho\lambda}(y)Y_\nu^{\mu a}(t)Y_\epsilon^{a\sigma}(v)N_{2\sigma\rho}^{\nu\epsilon}, \\ N_3 &= 4\int dydtdv\, c^{\rho\lambda}(y)Y_\nu^{\mu a}(t)Y_\epsilon^{a\sigma}(v)N_{3\sigma\rho\lambda\mu}^{\nu\epsilon}, \end{aligned} \quad \text{(B.19)}$$

with

$$\begin{aligned} N_1^{\nu\lambda\mu\epsilon} &= \Delta(v-y)\partial_t^\nu\Delta^{\lambda\mu}(y-t)\partial_v^\epsilon\Delta(t-v), \\ N_2^{\sigma\rho\nu\epsilon} &= \Delta^{\sigma\rho}(v-y)\partial_t^\nu\Delta(y-t)\partial_v^\epsilon\Delta(t-v), \\ N_3^{\sigma\rho\nu\lambda\mu\epsilon} &= \Delta^{\sigma\rho}(v-y)\partial_t^\nu\Delta^{\lambda\mu}(y-t)\partial_v^\epsilon\Delta(t-v). \end{aligned} \quad \text{(B.20)}$$

The integrals $N_1^{\nu\epsilon\lambda\mu}$, $N_2^{\sigma\rho\nu\epsilon}$ have the same divergent part as $M_2^{\mu\nu\sigma\rho}$. Thus,

$$\begin{aligned} N_1^{\nu\epsilon\lambda\mu} &= -\frac{1}{24}(a-1)\frac{i}{16\pi^2\epsilon}[g^{\mu\nu}g^{\lambda\epsilon} + g^{\nu\lambda}g^{\mu\epsilon} + g^{\mu\lambda}g^{\nu\epsilon}]\delta(y-v)\delta(t-y) + \mathrm{f.\,p.}, \\ N_2^{\sigma\rho\nu\epsilon} &= -\frac{1}{24}(a-1)\frac{i}{16\pi^2\epsilon}[g^{\sigma\rho}g^{\nu\epsilon} + g^{\nu\sigma}g^{\rho\epsilon} + g^{\sigma\epsilon}g^{\rho\nu}]\delta(y-v)\delta(t-y) + \mathrm{f.\,p.} \end{aligned} \quad \text{(B.21)}$$

Substituting $c^{\rho\lambda}Y_\nu^{\mu a}Y_\epsilon^{a\sigma} = (\rho + X^aX^a)X^bX^bg^{\rho\lambda}\delta_\nu^\mu\delta_\epsilon^\sigma$, we obtain

$$N_1 + N_2 = 8(a-1)\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^aX^a)X^bX^b + \mathrm{f.\,p.}$$

The integral $N_3^{\sigma\rho\nu\lambda\mu\epsilon}$ is of the form

$$N_3^{\sigma\rho\nu\lambda\mu\epsilon} = (a-1)^2)\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}e^{ip(y-v)+iq(t-y)}G_3^{\sigma\rho\nu\lambda\mu\epsilon},$$

where

$$G_3^{\sigma\rho\nu\lambda\mu\epsilon} = 4!\int_0^1 dx\int_0^x dy\int_0^y dz\int_0^z du\int \frac{d^dk}{(2\pi)^d}\frac{k^\sigma k^\rho k^\lambda k^\mu k^\nu k^\epsilon}{[k^2+\ldots]^5} + \mathrm{f.\,p.}$$

Using the value of the integral in appendix A and then noting that

$$g_{\rho\lambda}g_{\mu\nu}g_{\sigma\epsilon}[g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon} + \mathrm{perm.}] = 192$$

we get N_3, which reads

$$N_3 = 4(a-1)^2\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^aX^a)X^bX^b + \mathrm{f.\,p.}$$

Thus, the divergent part of $\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)$ is

$$\begin{aligned}
&div\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) = div\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)^{a=1}\\
&+6(a-1)\frac{i}{16\pi^2\epsilon}\int dxY_\mu^{ab}X^a\partial^\mu X^b\\
&-\frac{1}{2}(a-1)\frac{i}{16\pi^2\epsilon}\int dx\left\{6Y_\rho^{ab}X^b(b^{\rho a}+b^{a\rho}) - 24b^{ab}X^aX^b\right\}\\
&+\left\{24(a-1)+12(a-1)^2\right\}\frac{i}{16\pi^2\epsilon}\int dx(\rho + X^aX^a)X^bX^b. \qquad \text{(B.22)}
\end{aligned}$$

B.3 $\mathrm{Tr}(\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta\boldsymbol{D}_0^{-1}\delta)$

Finally, we calculate the divergent part of $\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)$. We have:

$$\begin{aligned}
&\mathrm{Sp}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) = (D_0^{-1})_{ab}(\delta)^{bc}(D_0^{-1})_{cd}(\delta)^{de}(D_0^{-1})_{ef}(\delta)^{fg}(D_0^{-1})_{gk}(\delta)^{ka}\\
&+4(D_0^{-1})_{ab}(\delta)^{bc}(D_0^{-1})_{cd}(\delta)^{de}(D_0^{-1})_{ef}(\delta)^{f\sigma}(D_0^{-1})_{\sigma\rho}(\delta)^{\rho a}\\
&+2(D_0^{-1})_{ab}(\delta)^{b\sigma}(D_0^{-1})_{\sigma\rho}(\delta)^{\rho e}(D_0^{-1})_{ef}(\delta)^{f\lambda}(D_0^{-1})_{\lambda\mu}(\delta)^{\mu a} + \mathrm{f.\,p.}, \qquad \text{(B.23)}
\end{aligned}$$

where we omitted terms, that produce finite integrals. Then

$$\begin{aligned}
&\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) - \mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)^{a=1} = 4P + 2Q + \mathrm{f.\,p.}\\
&= 4\int dxdydzdtdudvdrds\,\{\Delta(x-y)\Delta(z-t)\Delta(u-v)\Delta_{\sigma\rho}(r-s)\times\\
&\times\langle y|(\delta)^{ab}|z\rangle\langle t|(\delta)^{bc}|u\rangle\langle v|(\delta)^{c\sigma}|r\rangle\langle s|(\delta)^{\rho a}|x\rangle\}\\
&+2\int dxdydzdtdudvdrds\,\{-\Delta(x-y)\Delta(z-t)\Delta(u-v)\Delta_{\lambda\mu}(r-s)g_{\sigma\rho}\\
&-\Delta(x-y)\Delta_{\sigma\rho}(z-t)\Delta(u-v)\Delta(r-s)g_{\lambda\mu} + \Delta(x-y)\Delta_{\sigma\rho}(z-t)\Delta(u-v)\Delta_{\lambda\mu}(r-s)\}\times\\
&\times\langle y|(\delta)^{a\sigma}|z\rangle\langle t|(\delta)^{\rho b}|u\rangle\langle v|(\delta)^{b\lambda}|r\rangle\langle s|(\delta)^{\mu a}|x\rangle + \mathrm{f.\,p.} \qquad \text{(B.24)}
\end{aligned}$$

The integral P is of the form

$$P = 16\int dydtdvds\, Y_\mu^{ab}(y)Y_\nu^{bc}(t)Y_\lambda^{c\sigma}(v)Y_\epsilon^{\rho a}(s)P_{\sigma\rho}^{\mu\nu\lambda\epsilon} + \mathrm{f.\,p.},$$

where

$$P^{\mu\nu\lambda\epsilon\sigma\rho} = \partial_y^\mu\Delta(s-y)\partial_t^\nu\Delta(y-t)\partial_v^\lambda\Delta(t-v)\partial_s^\epsilon\Delta^{\sigma\rho}(v-s).$$

The integral $P^{\mu\nu\lambda\epsilon\sigma\rho}$ can be written as

$$P^{\mu\nu\lambda\epsilon\sigma\rho} = -(a-1)\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\frac{d^dk_4}{(2\pi)^d}\frac{k_1^\mu k_2^\nu k_3^\lambda k_4^\epsilon k_4^\sigma k_4^\rho}{k_1^2k_2^2k_3^2k_4^4}e^{-ik_1(s-y)-ik_2(y-t)-ik_3(t-v)-ik_4(v-s)}.$$

Introducing the new integration variables

$$(k_1,k_2,k_3,k_4)\mapsto(p,k,q,r):\quad p=k_1-k_2,\ q=k_3-k_2,\ r=k_4-k_3,\ k=k_2,$$

with Jacobian $J=1$, we get

$$P^{\mu\nu\lambda\epsilon\sigma\rho} = -(a-1)\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}e^{ip(y-s)+iq(s-t)+ir(s-v)}H^{\mu\nu\lambda\epsilon\sigma\rho}$$

where

$$H^{\mu\nu\lambda\epsilon\sigma\rho} = 4!\int_0^1 dx\int_0^x dy\int_0^y dz\int_0^z du\int\frac{d^dk}{(2\pi)^d}\frac{k^\sigma k^\rho k^\lambda k^\mu k^\nu k^\epsilon}{[k^2+\ldots]^5} + \mathrm{f.\,p.}$$

The latter integral has the same divergent part as $G_3^{\sigma\rho\nu\lambda\mu\epsilon}$. Thus,

$$H^{\mu\nu\lambda\epsilon\sigma\rho} = \frac{1}{8\Gamma(5)}\frac{i}{16\pi^2\epsilon}\frac{1}{24}[g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon}+\mathrm{perm.}]\delta(y-s)\delta(s-t)\delta(s-v) + \mathrm{f.\,p.}$$

Substituting $Y_\lambda^{c\sigma}Y_\epsilon^{\rho a} = -X^aX^c\delta_\lambda^\sigma\delta_\epsilon^\rho$ and noting that

$$g_{\sigma\lambda}g_{\rho\epsilon}[g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon}+\mathrm{perm.}] = 48g^{\mu\nu},$$

we obtain

$$P = 4(a-1)\frac{i}{16\pi^2\epsilon}\int dxg^{\mu\nu}Y_\mu^{ab}Y_\nu^{bc}X^aX^c + \mathrm{f.\,p.}$$

We write the second integral Q as

$$Q = Q_1+Q_2+Q_3+\mathrm{f.\,p.},$$

where

$$Q_1 = -16\int dydtdvds\, g_{\alpha\beta}Y_\sigma^{a\alpha}(y)Y_\rho^{\beta b}(t)Y_\nu^{b\lambda}(v)Y_\epsilon^{\mu a}(s)Q_{1\lambda\mu}^{\sigma\rho\nu\epsilon} + \mathrm{f.\,p.},$$

$$
\begin{aligned}
Q_2 &= -16\int dydtdvds\, g_{\alpha\beta}Y^{a\lambda}_{\sigma}(y)Y^{\mu b}_{\rho}(t)Y^{b\alpha}_{\nu}(v)Y^{\beta a}_{\epsilon}(s)Q^{\sigma\rho\nu\epsilon}_{2\lambda\mu} + \mathrm{f.\,p.}, \\
Q_3 &= 16\int dydtdvds\, Y^{a\sigma}_{\alpha}(y)Y^{\rho b}_{\beta}(t)Y^{b\lambda}_{\nu}(v)Y^{\mu a}_{\epsilon}(s)Q^{\alpha\beta\nu\epsilon}_{3\sigma\rho\lambda\mu} + \mathrm{f.\,p.},
\end{aligned} \tag{B.25}
$$

with

$$
\begin{aligned}
Q_1^{\sigma\rho\nu\epsilon\lambda\mu} &= \partial^{\sigma}_{y}\Delta(s-y)\partial^{\rho}_{t}\Delta(y-t)\partial^{\nu}_{v}\Delta(t-v)\partial^{\epsilon}_{s}\Delta^{\lambda\mu}(v-s), \\
Q_2^{\sigma\rho\lambda\mu\nu\epsilon} &= \partial^{\sigma}_{y}\Delta(s-y)\partial^{\rho}_{t}\Delta^{\lambda\mu}(y-t)\partial^{\nu}_{v}\Delta(t-v)\partial^{\epsilon}_{s}\Delta(v-s), \\
Q_3^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu} &= \partial^{\alpha}_{y}\Delta(s-y)\partial^{\beta}_{t}\Delta^{\sigma\rho}(y-t)\partial^{\nu}_{v}\Delta(t-v)\partial^{\epsilon}_{s}\Delta^{\lambda\mu}(v-s).
\end{aligned} \tag{B.26}
$$

The integrals $Q_1^{\sigma\rho\nu\epsilon\lambda\mu}, Q_2^{\sigma\rho\lambda\mu\nu\epsilon}$ have the same divergent part as $H^{\mu\nu\lambda\epsilon\sigma\rho}$. Since

$$
g_{\sigma\rho}g_{\lambda\nu}g_{\mu\epsilon}[g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon} + \mathrm{perm.}] = 192, \quad g_{\nu\epsilon}g_{\lambda\sigma}g_{\mu\rho}[g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon} + \mathrm{perm.}] = 192,
$$

we obtain

$$
Q_1 + Q_2 = 32(a-1)\frac{i}{16\pi^2\epsilon}\int dx X^a X^a X^b X^b + \mathrm{f.\,p.}
$$

The third integral $Q_3^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu}$ is

$$
Q_3^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu} = (a-1)^2\int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}e^{ip(y-s)+iq(s-t)+ir(s-v)}H_3^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu},
$$

where

$$
H_3^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu} = 5!\int_0^1 dx\int_0^x dy\int_0^y dz\int_0^z du\int_0^u dv\int\frac{d^dk}{(2\pi)^d}\frac{k^\alpha k^\beta k^\sigma k^\rho k^\nu k^\epsilon k^\lambda k^\mu}{[k^2+\ldots]^6} + \mathrm{f.\,p.}
$$

Performing necessary integrations, we get

$$
Q_3^{\alpha\beta\sigma\rho\nu\epsilon\lambda\mu} = (a-1)^2\frac{1}{16}\frac{i}{16\pi^2\epsilon}\frac{1}{120}[g^{\alpha\beta}g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon} + \mathrm{perm.}]\delta(y-s)\delta(s-t)\delta(s-v) + \mathrm{f.\,p.}
$$

Then we substitute $Y^{a\sigma}_{\alpha}Y^{\rho b}_{\beta}Y^{b\lambda}_{\nu}Y^{\mu a}_{\epsilon} = X^aX^aX^bX^b\delta^{\sigma}_{\alpha}\delta^{\rho}_{\beta}\delta^{\lambda}_{\nu}\delta^{\mu}_{\epsilon}$. Since

$$
g_{\sigma\alpha}g_{\rho\beta}g_{\lambda\nu}g_{\mu\epsilon}[g^{\alpha\beta}g^{\mu\nu}g^{\sigma\rho}g^{\lambda\epsilon} + \mathrm{perm.}] = 1920,
$$

we obtain

$$
Q_3 = 16(a-1)^2\frac{i}{16\pi^2\epsilon}\int dx X^aX^aX^bX^b + \mathrm{f.\,p.}
$$

Hereby, the divergent part of $\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)$ reads

$$
div\,\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta) = div\,\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta D_0^{-1}\delta)^{a=1}
$$

$$+16(a-1)\frac{i}{16\pi^2\epsilon}\int dx g^{\mu\nu}Y^{ab}_{\mu}Y^{bc}_{\nu}X^aX^c$$
$$+\left\{64(a-1)+32(a-1)^2\right\}\frac{i}{16\pi^2\epsilon}\int dx X^aX^aX^bX^b \tag{B.27}$$

B.4 Summation over flavour indices

Thus, the divergent part of the one-loop functional in the arbitrary gauge is given by the expression

$$div Z_{\text{one loop}} = div Z^{a=1}_{\text{one loop}} + \frac{1}{2}(1-a)\frac{1}{16\pi^2\epsilon}\int dx \left\{-2b^{\rho a}\partial_\rho X^a + \frac{1}{4}g_{\sigma\rho}b^{\sigma a}b^{\rho a}\right.$$
$$\left. +2Y^{ab}_{\mu}X^a\partial^\mu X^b - Y^{ab}_{\rho}X^b(b^{\rho a}+b^{a\rho}) + 4b^{ab}X^aX^b - 4g^{\mu\nu}Y^{ab}_{\mu}Y^{bc}_{\nu}X^aX^c\right\} \tag{B.28}$$
$$+\left\{2(a-1)+(a-1)^2\right\}\frac{1}{16\pi^2\epsilon}\int dx\left\{\frac{1}{4}(\rho+X^aX^a)^2 - 2(\rho+X^aX^a)X^bX^b + 4X^aX^aX^bX^b\right\}$$

For simplification of the result, we have to sum over flavor indices. This is done with help of the formula, which follows from the completeness relation for the generators λ^a of SU(N):

$$\sum_a \langle A\lambda^a\rangle\langle B\lambda^a\rangle = 2\langle AB\rangle - \frac{2}{N}\langle A\rangle\langle B\rangle. \tag{B.29}$$

We further set $Q_R = Q_L = Q$. Since $H_L = \langle Q_R - Q_L\rangle = \langle Q - Q\rangle = 0$,

$$X^aX^a = \frac{F_0^2}{16}\langle H_L\lambda^a\rangle\langle H_L\lambda^a\rangle = \frac{F_0^2}{8}\langle H_L^2\rangle.$$

Then, $\rho + X^aX^a = \frac{1}{2}F_0^2\langle H_L^2\rangle$ and the second integral in Eq. (B.28) is equal to zero:

$$\frac{1}{4}(\rho+X^aX^a)^2 - 2(\rho+X^aX^a)X^bX^b + 4X^aX^aX^bX^b = \left(\frac{1}{16}-\frac{1}{8}+\frac{1}{16}\right)F_0^2\langle H_L^2\rangle = 0.$$

From the definition of the matrix b it follows that

$$\begin{aligned} b^{\rho a} &= X^{\rho c}_{\mu}\Gamma^{\mu ca} + \Lambda^{\rho a} - \partial^\mu X^{\rho a}_{\mu}, \quad b^{\rho a} + b^{a\rho} = 2(X^{\rho c}_{\mu}\Gamma^{\mu ca} + \Lambda^{\rho a}), \\ b^{ab} &= \Gamma^{ac}_{\mu}\Gamma^{\mu cb} + 4X^aX^b + \Lambda^{ab} - \partial^\mu\Gamma^{ab}_{\mu}. \end{aligned} \tag{B.30}$$

Then,

$$4b^{ab}X^aX^b - 4g^{\mu\nu}Y^{ab}_{\mu}Y^{bc}_{\nu}X^aX^c = 4X^aX^b\left\{\frac{1}{2}\langle[\Delta_\mu,\lambda^a][\Delta^\mu,\lambda^b]\rangle + \frac{1}{4}\langle\{\lambda^a,\lambda^b\}\sigma\rangle\right.$$
$$\left. -\frac{C}{8F_0^2}(\langle[H_R+H_L,\lambda^a][H_R-H_L,\lambda^b] + a\leftrightarrow b\rangle)\right\} \tag{B.31}$$

The matrix element $b^{\rho a}$ can be written as

$$b^{\rho a} = -\frac{1}{2} F_0 \langle ([H_R, \Delta^\rho] + D^\rho H_L)\, \lambda^a \rangle,$$

since $X^{\rho b}_\mu \Gamma^{\mu b a} = -\frac{1}{4} F_0 \langle \lambda^a [\Gamma^\rho, H_L] \rangle$. Using Eq. (B.29), we obtain:

$$\begin{aligned}
& b^{\rho a} \partial_\rho X^a = \frac{1}{4} F_0^2 \langle \partial_\mu H_L ([H_R, \Delta^\mu] + D^\mu H_L) \rangle, \\
& g_{\sigma\rho} b^{\sigma a} b^{\rho a} = \frac{1}{2} F_0^2 \langle ([H_R, \Delta_\mu] + D_\mu H_L)([H_R, \Delta^\mu] + D^\mu H_L) \rangle, \\
& \Gamma^{ab}_\mu X^a \partial^\mu X^b = -\frac{1}{8} F_0^2 \langle \partial^\mu H_L [\Gamma_\mu, H_L] \rangle, \\
& \Gamma^{ab}_\rho X^b X^c \Gamma^{\rho c a} = -\frac{1}{8} F_0^2 \langle [\Gamma^\mu, H_L][\Gamma_\mu, H_L] \rangle, \\
& \Gamma^{ab}_\rho X^b \Lambda^{\rho a} = \frac{1}{4} F_0^2 \langle [\Gamma_\mu, H_L] \left([H_R, \Delta^\mu] + \frac{1}{2} D^\mu H_L \right) \rangle, \\
& X^a X^b \langle [\Delta_\mu, \lambda^a][\Delta^\mu, \lambda^b] \rangle = \frac{1}{4} F_0^2 \langle [H_L, \Delta_\mu][H_L, \Delta^\mu] \rangle, \\
& X^a X^b \langle \{\lambda^a, \lambda^b\} \sigma \rangle = \frac{1}{2} F_0^2 \langle H_L^2 \sigma \rangle, \\
& X^a X^b (\langle [H_R + H_L, \lambda^a][H_R - H_L, \lambda^b] + a \leftrightarrow b \rangle) = \frac{1}{8} F_0^2 \langle [H_R + H_L, H_R - H_L]^2 \rangle
\end{aligned} \quad \text{(B.32)}$$

We also note that $D_\mu H_L$ can be written as

$$D_\mu H_L = [H_R, \Delta_\mu] + G_\mu,$$

where

$$G^\mu = u^+ c^\mu_R Q u - u c^\mu_L Q u^+.$$

After substitution of these formulas into Eq. (B.28), the divergent part of the one-loop functional takes the form

$$\begin{aligned}
div Z_{\text{one loop}} = div Z^{a=1}_{\text{one loop}} + \frac{1}{4} F_0^2 (1-a) \frac{1}{16\pi^2 \epsilon} \int dx \Big\{ \langle [H_L, \Delta_\mu]^2 \rangle - \langle [H_R, \Delta_\mu]^2 \rangle \\
+ \langle H_L^2 \sigma \rangle - 2 \langle [H_R, \Delta_\mu] G^\mu \rangle - \frac{3}{4} \langle G_\mu G^\mu \rangle - \frac{1}{8} F_0^2 Z \langle [H_R + H_L, H_R - H_L]^2 \rangle \Big\},
\end{aligned} \quad \text{(B.33)}$$

where $Z = C/F_0^4$.